Revista da Estatística UFOP, Vol III(3), 2014, ISSN 2237-8111
Edição especial: 59ª Reunião Anual da Regional Brasileira da Sociedade Internacional de Biometria - RBRAS
Trabalhos apresentados na versão poster.

Intensity estimation for a compound Poisson driven SDE

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1 Introduction

Compound Poisson processes are stochastic processes that can model random shocks at random times. The random times come from a Poisson point process on the real line and the magnitude of the shocks come from a set of real random variables. They have been successfully used in finance and insurance models. Stochastic Differential Equations have been used in many fields of science and technology, having applications in engineering stochastic control, economics, finance and population biology, for example. Inference for a Poisson point process is done by the estimation of its intensity function, as this function characterizes the Poisson point process. In case of homogeneous Poisson processes, these are constant functions and a real parameter is sufficient to have the estimation done. See [1], [3], [4], [5] and [6] for point processes and their estimation and [7] and [8] for stochastic differential equations and some of their applications. In this work we will study a stochastic differential equation driven by a compound Poisson process. More precisely, we will be interested in the following model:

$$\frac{\mathrm{d}X}{\mathrm{d}t} + \alpha(t)X = \sum_{i=1}^{\mathbf{N}(0,t]} Y_i \tag{1}$$

where $\mathbf{N}$ is an homogeneous Poisson process with intensity λ and $Y_i \sim Y$ for all $i \in \mathbb{N}^*$. We assume that $\{\mathbf{N}, Y_i : i \in \mathbb{N}^*\}$ is a probabilistically independent set. The random variable Y is such that $\mathbb{E}Y = 0$ and $\mathbb{V}ar(Y) = \sigma^2 < \infty$. The initial condition $X(0)$ is assumed to be a deterministic value. This SDE can be used to model capital growth at time varying interest rates in an economic environment with random shocks which are not proportional to the capital under interests. Another application of this SDE is in modeling population growth under random shocks that can either increase or decrease population at random times.

We will focus on the estimation of the Poisson intensity in three different information settings: First, based on the knowledge of a set of N independent trajectories of the process, $\{X_i(t) : 0 \leq t \leq T, 1 \leq i \leq N\}$, then based on N independent evaluations of trajectories of the process at a fixed time τ, i.e., on $\{X_i(\tau) : 1 \leq i \leq N\}$, and, finally, based on a set of N independent evaluations of trajectories of the process at possibly different times τ_i, i.e., on $\{X_i(\tau_i) : 1 \leq i \leq N\}$, and on the common initial value $X(0)$. In these three inferential situations, we construct unbiased, consistent and asymptotically normal estimators, $\hat{\lambda}_1$, $\hat{\lambda}_2$, and $\hat{\lambda}_3$, respectively, of the Poisson intensity.

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2 The solutions of the compound Poisson driven SDE

The solutions to (1) are the following:

$$X(t) = e^{-\int_0^t \alpha(z)\mathrm{d}z} \left(\int_0^t e^{\int_0^z \alpha(w)\mathrm{d}w} \sum_{i=1}^{\mathbf{N}(0,z]} Y_i \mathrm{d}z + X(0) \right) \tag{2}$$

where the dependence on the initial condition is already present.

3 Estimator construction

3.1 Properties of the solutions

The expectation, variance and centered fourth moment functions, when $\mu_4 = \mathbb{E}Y^4 < \infty$, associated to the stochastic process $X(t)$ are given by the following theorem

Theorem 1: *Let $X(t)$ be the solution to (1). Then, denoting $\mathcal{A}(t) = e^{\int_0^t \alpha(z)\mathrm{d}z}$, we have*

$$\mathbb{E}X(t) = \mathcal{A}(t)^{-1} X(0), \tag{3}$$

$$\mathbb{V}arX(t) = \lambda\sigma^2 \mathcal{A}(t)^{-2} \int_0^t \int_0^t \mathcal{A}(z)\mathcal{A}(s)(z \wedge s)\mathrm{d}z\mathrm{d}s, \tag{4}$$

and

$$\mathbb{M}_4 X(t) = \mathbb{E}(X(t) - \mathbb{E}X(t))^4 =$$

$$\lambda^2\sigma^4 \mathcal{A}(t)^{-4} \int_0^t \int_0^t \int_0^t \int_0^t \mathcal{A}(z_1)\mathcal{A}(z_2)\mathcal{A}(z_3)\mathcal{A}(z_4) z_{(1)} \left(2z_{(2)} + z_{(3)} + \frac{\mu_4}{\lambda\sigma^4} \right) \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4, \tag{5}$$

where $z_{(1)} \leq z_{(2)} \leq z_{(3)} \leq z_{(4)}$ is an ordering of z_1, z_2, z_3 and z_4.

3.2 The estimators

3.2.1 Estimation with known trajectories

The following estimator is based on the number of occurrences of a Poisson process on the interval $[0,T]$. We will assume that $\mathbb{P}(Y = 0) = 0$. Since we know the whole trajectories, X_i, we also know the times of the occurrences for each of the independent copies, $\mathbf{N}_i$, of the Poisson process $\mathbf{N}$ for $1 \leq i \leq N$. These times are those for which the derivative of the solution, X_i, presents a discontinuity. Let us denote the number of occurrences of $\mathbf{N}_i$ in $[0,T]$ by $\mathfrak{O}_i$.

The set $\{\mathfrak{O}_i : 1 \leq i \leq N\}$ is a random sample of a Poisson distributed random variable with mean λT. Thus,

$$\hat{\lambda}_1 = \frac{1}{NT} \sum_{i=1}^{N} \mathfrak{O}_i \tag{6}$$

is a natural choice for an estimator of λ.

3.2.2 Estimation with known values of trajectories at a fixed time

Let

$$\mathcal{K}(t) = \sigma^2 \mathcal{A}(t)^{-2} \int_0^t \int_0^t \mathcal{A}(z)\mathcal{A}(s)(z \wedge s)\mathrm{d}z\mathrm{d}s. \tag{7}$$

Based on (4) we define the estimator of the intensity by:

$$\hat{\lambda}_2 = \frac{1}{\mathcal{K}(\tau)} \frac{\sum_{i=1}^{N} \left(X_i(\tau) - \overline{X(\tau)} \right)^2}{N-1}. \tag{8}$$

where $\overline{X(\tau)} = \frac{\sum_{i=1}^{N} X_i(\tau)}{N}$.

3.2.3 Estimation with known values of trajectories at distinct times and known initial condition

Also based on (4) and on a convenient transformation of variables we define the following intensity estimator

$$\hat{\lambda}_3 = \frac{1}{N} \sum_{i=1}^{N} \frac{\left(X_i(\tau_i) - X(0)\mathcal{A}^{-1}(\tau_i) \right)^2}{\mathcal{K}(\tau_i)}. \tag{9}$$

4 Main results

Now we present the main results concerning the properties of $\hat{\lambda}_1$, $\hat{\lambda}_2$, and $\hat{\lambda}_3$.

Theorem 2: *Let $X_i(t)$, $1 \leq i \leq N$, be a solution to the Poisson driven SDE: $\frac{\mathrm{d}X_i}{\mathrm{d}t} + \alpha(t)X_i = \sum_{j=1}^{\mathbf{N}_i(0,t]} Y_{i,j}$, where $\{\mathbf{N}_i : 1 \leq i \leq N\}$ is an i.i.d. set of Poisson processes with intensity λ, the initial condition is the same for all i, $X_i(0) = X(0)$, and $\alpha : [0,T] \to \mathbb{R}$ is a known integrable function. Assume that $\{Y_{i,j} : i,j \in \mathbb{N}\}$ is an i.i.d. set of random variables distributed like Y with $\mathbb{E}Y = 0$ and $\mathbb{V}ar(Y) = \sigma^2 < \infty$. Assume that σ is known and that $\{\mathbf{N}_i, Y_{i,j} : i,j \in \mathbb{N}\}$ is an independent set of random elements. (These assumptions will be called [$\mathcal{H}$] from here on).*

Suppose we know the set of solutions $\{X_i(t) : 0 \leq t \leq T,\ 1 \leq i \leq N\}$. Let $\mathfrak{D}_i$ be the number of discontinuities of the derivative of X_i on $[0,T]$.

Then $\hat{\lambda}_1 = \frac{1}{NT} \sum_{i=1}^{N} \mathfrak{D}_i$ is an unbiased, consistent and asymptotically normal estimator of λ, and we have the following convergence in distribution: $\sqrt{TN}(\hat{\lambda}_1 - \lambda) \rightsquigarrow \mathcal{N}(0,1)$, as $N \to \infty$.

From here on we will use the following notation:

$$\mathfrak{R}(t) = \frac{\int_0^t \int_0^t \int_0^t \int_0^t \mathcal{A}(z_1)\mathcal{A}(z_2)\mathcal{A}(z_3)\mathcal{A}(z_4) z_{(1)} \left(2z_{(2)} + z_{(3)} + \frac{\mu_4}{\lambda\sigma^4} \right) \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4}{\left(\int_0^t \int_0^t \mathcal{A}(z_1)\mathcal{A}(z_2) z_{(1)} \mathrm{d}z_1\mathrm{d}z_2 \right)^2}. \tag{10}$$

Theorem 3: *Assume condition [$\mathcal{H}$]. Additionally, suppose that $\mathbb{E}Y^4 < \infty$. Let τ be a fixed time in $(0,T]$. Suppose we know the set of evaluations of the solutions X_i at this time only, i.e., we only know $\{X_i(\tau) : 1 \leq i \leq N\}$. Then $\hat{\lambda}_2 = \frac{1}{\mathcal{K}(\tau)} \frac{\sum_{i=1}^N \left(X_i(\tau) - \overline{X(\tau)}\right)^2}{N-1}$, where $\mathcal{K}(t) = \sigma^2 \mathcal{A}(t)^{-2} \int_0^t \int_0^t \mathcal{A}(z)\mathcal{A}(s)(z \wedge s)\mathrm{d}z\mathrm{d}s$ and $\mathcal{A}(t) = e^{\int_0^t \alpha(z)\mathrm{d}z}$, is an unbiased, consistent and asymptotically normal estimator of λ. We have the following convergence in distribution:*

$$\sqrt{N}(\hat{\lambda}_2 - \lambda) \rightsquigarrow \mathcal{N}\left(0, \lambda^2 \left(\mathfrak{R}(\tau) - 1\right)\right). \tag{11}$$

Theorem 4: *Assume condition [$\mathcal{H}$] and $\mathbb{E}Y^4 < \infty$. Let τ_i, $1 \leq i \leq N$, be fixed times in $(0,T]$. Suppose we know the set of evaluations $\{X_i(\tau_i) : 1 \leq i \leq N\}$ and the initial condition $X(0)$ which is assumed to be the same for all trajectories X_i, i.e., for all i, $X_i(0) = X(0)$. Then $\hat{\lambda}_3 = \frac{1}{N} \sum_{i=1}^N \frac{\left(X_i(\tau_i) - X(0)\mathcal{A}^{-1}(\tau_i)\right)^2}{\mathcal{K}(\tau_i)}$, where $\mathcal{K}(t) = \sigma^2 \mathcal{A}(t)^{-2} \int_0^t \int_0^t \mathcal{A}(z)\mathcal{A}(s)(z \wedge s)\mathrm{d}z\mathrm{d}s$ and $\mathcal{A}(t) = e^{\int_0^t \alpha(z)\mathrm{d}z}$, is an unbiased estimator of λ. Moreover, its variance is given by:*

$$\mathbb{V}ar(\hat{\lambda}_3) = \lambda^2 \frac{1}{N^2} \sum_{i=1}^N \left(\mathfrak{R}(\tau_i) - 1\right). \tag{12}$$

We always have $\mathfrak{R}(\tau_i) > 1$. Now, letting $N \to \infty$, if the set of times τ_i is bounded away from zero by a positive constant δ, then we will also have $\mathfrak{R}(\tau_i) \leq \mathrm{C}(\delta, T) < \infty$. Under this assumption $\hat{\lambda}_3$ is a consistent estimator of λ. If, in addition, we assume that $\mathbb{E}Y^6 < \infty$ then $\hat{\lambda}_3$ is also asymptotically normal and we have:

$$\sqrt{N}(\hat{\lambda}_3 - \lambda) \rightsquigarrow \mathcal{N}\left(0, \lambda^2 \frac{1}{N} \sum_{i=1}^N \left(\mathfrak{R}(\tau_i) - 1\right)\right). \tag{13}$$

5 Final remarks

Under the hypothesis of Theorem 4, we have two constants, $c(\delta, T)$ and $C(\delta, T)$ such that, for all i, $1 < c(\delta, T) \leq \mathfrak{R}(\tau_i) < C(\delta, T)$ and, consequently, $\sqrt{N}(\hat{\lambda}_3 - \lambda) \rightsquigarrow \mathcal{N}\left(0, \lambda^2 \mathfrak{v}\right)$ for some $\mathfrak{v} \in [c(\delta, T) - 1, C(\delta, T) - 1]$.

In case we know the whole solutions, which is our first estimation setting, we can relax the assumption on the initial condition and assume different initial conditions $X_i(0) = c_i \in \mathbb{R}$ for each i from 1 to N. As a matter of fact, even if the initial condition is assumed to be be a random variable such that $\{X(0), \mathbf{N}, Y_i : i \in \mathbb{N}\}$ is an independent set, $\hat{\lambda}_1$ will still be unbiased, consistent and asymptotically normal. Note that the initial condition does not change the number of discontinuities of the derivatives of the trajectories. This number is exactly the number of occurrences of the Poisson process which is all we need to estimate λ.

We observe that since it is possible to obtain expressions for the variance of the estimators $\hat{\lambda}_1$, $\hat{\lambda}_2$, and $\hat{\lambda}_3$ we can construct confidence intervals for λ.

In practice, given that we often have to deal with small sample sizes, it is expected that all three estimators will not perform very well if the product of the intensity by the time span is too small. If the sample size can not be increased, a way to obtain more reliable estimates is to measure the solutions for as greater values of T, τ or τ_i as possible. Concerning $\hat{\lambda}_1$, very small values of Y_i may bring difficulties in finding discontinuities on the derivatives. This will introduce a negative bias since the actual number of occurrences will be greater than the measured one.

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