Articles Section
2024, Volume 6, e2024007
Pages 1–18
10.33532/revemop.e2024007

revemop
ISSN 2596-0245

Exponential Functions and their Derivatives in the light of Conceptual Image and Conceptual Definition

As Funções Exponenciais e suas Derivadas à luz da Imagem Conceitual e da Definição Conceitual

Funciones Exponenciales y sus Derivadas a la luz de la Imagen Conceptual y la Definición Conceptual

Saulo Macedo de Oliveira[1]
Rieuse Lopes[2]

Abstract

This article is an excerpt from a dissertation carried out with students studying Differential and Integral Calculus in the first term of the Information Systems course at the State University of Montes Claros and is based on the theoretical constructs of symbolic interactionism and advanced mathematical thinking, specifically the conceptual image and the conceptual definition of exponential functions. It aimed to understand how and in what way mathematical definitions are used in discussions held in interactions between students and professors during the students' presentation of a seminar. From the analysis of the data, it was possible to observe divergences between the images and the conceptual definitions of exponential functions and their derivatives. We concluded that the students had made progress in understanding the formal definitions from the advanced mathematical thinking perspective.
Keywords: Conceptual Definition. Exponential Functions and their Derivatives. Conceptual Image. Symbolic Interactionism. Advanced Mathematical Thinking.

Resumo

Este artigo é um recorte de uma dissertação realizada com acadêmicos da disciplina Cálculo Diferencial e Integral, do 1º período do curso Sistemas de Informação da Universidade Estadual de Montes Claros, e fundamenta-se nos construtos teóricos do Interacionismo Simbólico e do Pensamento Matemático Avançado, especificamente a imagem conceitual e a definição conceitual de funções exponenciais. Teve como objetivo compreender como e de que forma as definições matemáticas são empregadas em discussões realizadas nas interações entre alunos e professores durante a apresentação de um seminário realizado pelos acadêmicos. A partir da análise dos dados, foi possível observar divergências entre as imagens e as definições conceituais de funções exponenciais e suas derivadas. Concluímos que houve um avanço, por parte dos acadêmicos, na compreensão das definições formais na concepção do Pensamento Matemático Avançado.
Palavras-chave: Definição Conceitual. Funções Exponenciais e suas Derivadas. Imagem Conceitual. Interacionismo Simbólico. Pensamento Matemático Avançado.

Resumen

Este artículo es un extracto de una disertación realizada con alumnos que cursan Cálculo Diferencial e Integral en el primer cuatrimestre del curso de Sistemas de Información de la Universidad Estadual de Montes Claros, y se basa en los constructos teóricos del Interaccionismo Simbólico y del Pensamiento Matemático Avanzado, específicamente la imagen conceptual y la definición conceptual de funciones exponenciales. El objetivo fue comprender cómo y de qué manera se utilizan las definiciones matemáticas en las discusiones mantenidas en las interacciones entre alumnos y profesores durante la presentación de un seminario por parte de los alumnos. A partir del análisis de los datos, fue posible observar divergencias entre las imágenes y las definiciones conceptuales de las funciones exponenciales y

1 Mestrando do Programa de Pós-Graduação em Educação, na linha de pesquisa Educação Matemática, da Universidade Estadual de Montes Claros (UNIMONTES). Licenciado em Matemática pela Universidade Estadual de Montes Claros (UNIMONTES), Montes Claros, Minas Gerais, Brasil. E-mail: saulomacedo308@gmail.com

2 Doutora em Educação Matemática pela Pontifícia Universidade Católica de São Paulo (PUC-SP). Professora do Departamento de Ciências Exatas da Universidade Estadual de Montes Claros (UNIMONTES), Montes Claros, Minas Gerais, Brasil. E-mail: rieuse.lopes@unimontes.br.

sus derivadas. Concluimos que los alumnos habían progresado en la comprensión de las definiciones formales en el concepto de Pensamiento Matemático Avanzado.
Palabras clave: Definición conceptual. Funciones exponenciales y sus derivadas. Imagen Conceptual. Interaccionismo Simbólico. Pensamiento Matemático Avanzado.

1. First Considerations

Dropouts and failures in the Integral and Differential Calculus (IDC) subject are commonplace in many higher education courses in Brazil and around the world. This subject, known for its complexity and mathematical abstraction, is often an obstacle for students, especially those who do not have a solid foundation in mathematics or who face difficulties adapting to higher education (Macêdo, GREGOR, 2020).

In the Brazilian context, students usually face difficulties in courses such as engineering, physics, biology, information systems, teaching degrees in mathematics, and majors in economic and social sciences, in which the Integral and Differential Calculus subject is fundamental. As Almeida, Barbosa, Musmanno, and Souza (2023) and Oliveira (2023) state, several reasons concur for students' dropout and failure, including lack of preparation, challenging teaching methodologies, lack of adequate academic support, and even emotional aspects, such as anxiety and self-confidence.

According to Guio and Barcellos (2021), many Brazilian students may have had a poor basic mathematics education, contributing to their difficulty when dealing with advanced calculus concepts. Another aspect concerns how the subject is conveyed: emphasis on solving practical problems can represent additional challenges. Also, a high percentage of students leave or fail this subject. This can reflect the difficulties mentioned or indicate problems in Calculus teaching and learning processes that may be related to the teaching methodology.

Several reasons contribute to failure and dropout in the IDC subject. One of the most important is students' lack of background in the subject content. Many students begin higher education with gaps in basic mathematics, especially in subjects such as algebraic equations and functions. Another reason that contributes to failure is the teaching methodology adopted. The IDC subject is usually presented in an abstract and decontextualized way, which makes learning more difficult for students.

In the Brazilian scenario, discussions about teaching practices, the implementation of innovative pedagogical methods, and the offer of more effective support resources are relevant topics to improving the success rate of students in this subject and, consequently, in their higher education courses.

Amancio and Sanzovo (2020) have identified these difficulties and developed, tested, and evaluated new methodologies from different perspectives and in different contexts, with or without the use of technological resources, that can contribute to reducing students' difficulties and assisting in the teaching process, with a positive impact on the understanding of IDC content and a consequent reduction in failure and dropout rates in the subject.

Rosa, Alvarenga, and Santos (2019) show that difficulties still exist. These authors present investigations that pointed out students' learning difficulties in the IDC subject and the high failure rate, highlighting "teacher's methodology as one of the problems faced during training" (Silva; Nascimento; Vieira, 2017, p. 13).

Therefore, this question inspired us to plan and apply activities to analyze and interpret the graphical relationships between exponential functions and their derivatives, as well as the properties of these functions, using graphical representation software. In this sense, we developed research to understand how and in what way mathematical definitions are used in dialogues between the students and the professors during presentations in a seminar, whose studies highlighted graphical and algebraic representations of exponential functions and their derivatives, elaborated using GeoGebra. Using software in teaching is relevant because, as Oliveira and Lopes (2023, p. 2) point out, "young people are currently part of a technological and multifaceted environment, which can be explored in favor of the learning process."

The article is structured as follows: in the next section, the theoretical framework underpinning this work will be explained, bringing together authors who research symbolic interactionism and advanced mathematical thinking. Afterward, the methodological path of this work will be discussed. The penultimate section addresses the analysis and discussions of the collected data; the last section presents the final considerations.

2. Advanced Mathematical Thinking and Symbolic Interactionism: a theoretical framework

In this section, we present a synthesis of the main characteristics of the theories underlying this work: advanced mathematical thinking, the stipulated, extracted, conceptual, formal conceptual, personal conceptual definition, and symbolic interactionism. We also highlight the idiosyncrasies of advanced mathematical thinking based on the theoretical notions of conceptual image and conceptual definition.

2.1. Advanced Mathematical Thinking

There are several approaches to the study of teaching and learning processes, and generally, the discussions focus on the type of learner one wants to educate. Therefore, teaching and learning concepts are essential, as they guide educators concerned with content, teaching methodologies, and mathematical thinking students develop in the learning process. Research has examined the evolution of students' mathematical thinking from elementary school to higher education (Tall, 1991; Vinner, 1991; Domingos, 2006).

In line with Domingos (2006), we affirm that the difficulties and the high dropout rate in mathematics can be attributed to students' low understanding of mathematical concepts. Due to the complexity inherent in understanding mathematical concepts, such as functions and derivatives, we seek a theoretical framework that helps us analyze, describe, and explain how university students express their understanding of these concepts.

Tall (1995) organized mathematical thinking from a cognitive perspective through three elements of human activity: capture as input, reasoning as internal processing, and execution as output. This allows us to consider mathematical activities such as perceiving objects, reasoning, and performing actions on them. Tall (1995) considers that mathematical thinking begins by capturing objects from the external world and the actions carried out on them. This thinking also develops simultaneously through processes that inspire creative reasoning based on the formal definition and systematic demonstration of mathematical concepts. As reasoning develops and becomes more complex, actions on these objects lead to advanced mathematical reasoning, which involves using cognitive structures produced by various mathematical activities. Thus, both elementary mathematical reasoning and advanced mathematical reasoning refer to how information is processed internally.

Tall (1995) distinguishes elementary mathematics, in which objects are characterized, from advanced mathematics, in which objects are defined rigorously. In elementary mathematics, the properties of objects are described from observation, while in advanced mathematics, they are deduced from definitions.

According to Tall (1991, p. 3), the characterization of the activity cycle in advanced mathematical thinking (AMT) leads "from the productive attitude of considering the contextualization of a problem in a mathematical investigation to the productive formulation of conjectures and the final stage of refinement and demonstration."

According to Dreyfus (1991), advanced mathematical thinking consists of an extensive series of processes that interact with each other, such as representation and abstraction. There are other processes besides these two, such as classifying, conjecturing, inducing, analyzing, and formalizing. However, it is through the two main processes that we move from one level of thought to another.

According to the author, these processes can be found in both elementary mathematical thinking and advanced mathematical thinking, as there are topics in basic mathematics that can be treated in an advanced way, just as there is elementary thinking about advanced topics, as the distinction is in the complexity of how the processes present in each of them are treated and managed. Thus, Dreyfus (1991) highlights the importance of the teacher knowing these processes because it facilitates understanding students' difficulties.

Regarding the definition of AMT and the discussions produced about its nature and development, we found in the specialized literature that "over time, there has been a diversity of opinions expressed on this topic, and there is no definition of AMT that is unanimously accepted" (Mamona-Downs; Downs, 2008, p. 155). These authors believe that the simplest option is that the AMT understands the cognitive needs to address mathematical content associated with domains usually covered at university.

According to Vinner (1991), when faced with a word associated with a mathematical concept, the individual evokes a mental representation of the concept in memory. This representation is called conceptual image. Tall and Vinner (1981) define the conceptual image as the totality of what is associated with the concept in the individual's mind, including mental images, processes, and properties. In this sense, these authors consider that:

> The conceptual image is something non-verbal associated in our mind with the name of the concept. It can be a visual representation of the concept if it has visual representations. It can also be a collection of impressions or experiences. The visual representations, mental pictures, impressions, and experiences associated with the name of the concept can be translated into verbal forms. But it is essential to remember that these verb forms are not the first thing

> evoked in our memory. They happen at a later stage. [...] When you hear the word "function," on the other hand, you may remember the expression "y = f(x)", you may visualize the graph of a function, you may think about specific functions like $y = x^2$ or y = sin(x), y = ln(x) etc. From what we have said, it is clear that a conceptual image can only be discussed in relation to a specific individual. Furthermore, the same individual could react differently to a certain term (concept name) in different situations. In Tall and Vinner (1981), the term "evoked conceptual image" is introduced to describe the part of the memory evoked in a given context. This is not necessarily all a certain individual knows about a certain notion. (VINNER, 1991, p. 6).

On the other hand, conceptual definition is the precise and non-redundant explanation of a concept through words. Tall and Vinner (1981) distinguish between formal conceptual definition, which is the exact and rigorous explanation, and personal conceptual definition, the verbal understanding of a person's formal definition. The conceptual definition, generally used for developing mathematical concepts in higher education, encompasses the formal and personal conceptual definitions. In this sense, Tall and Vinner emphasize that:

> The conceptual definition (if the individual possesses it) is entirely different. We consider the conceptual definition a form of words specifying that concept. It can be learned by an individual mechanically or in a more meaningful way, relating it to the concept as a whole to a greater or lesser extent. It can also be a personal reconstruction made by the student based on a definition. It constitutes a form of words that the student uses to explain their conceptual image (evoked). Whether the conceptual definition is given to the student or constructed by him/herself, he/she may vary it from time to time. In this sense, a personal conceptual definition may differ from a formal conceptual definition, the latter being a conceptual definition accepted by the mathematical community in general. (1981, p. 152, our translation)

Referencing the work of Tall and Vinner (1981), Meyer (2003) considers that the conceptual definition can also be constituted from the "personal reconstruction of the definition of a concept, without necessarily having coinciding meanings. In this case, the conceptual definition is the verbal form the student uses to specify his/her (evoked) conceptual image" (Meyer, 2003, p. 6).

2.2. Stipulated, Extracted, Conceptual, Formal Conceptual, and Personal Conceptual Definition

Considering the objective of this research, which is to understand how and in what way students use mathematical definitions in the graphical and algebraic representations of functions and their derivatives, we are based on these theoretical contributions to apply this foundation in the analysis of oral expressions and written by students when they use both formal and personal conceptual definitions. In addition to these authors, the analysis was also based on studies by Edwards and Ward (2004), which corroborate Tall and Vinner (1981) in understanding mathematical definitions but consider that conceptual definitions can be stipulated or extracted.

According to Edwards and Ward (2008), the authors state that stipulated definitions are an "explicit and self-conscious construction of the relationship of meaning between a word and some object, the act of assigning to an object a name (or a name to an object)." (Edwards; Ward, 2008, p. 224). A stipulated mathematical definition is a definition whose meanings in relation to the concept are designated or stipulated by the mathematical community and communicated by these symbols, i.e., by the formal definition.

In turn, the extracted definitions refer to concepts whose use in a variety of specific contexts allows an extraction or attribution of meanings to these concepts, which are referenced by their definitions. They are "definitions based on real examples, definitions extracted from a body of evidence." (Edwards; Ward, 2008, p. 224, our translation). According to the authors, most "everyday language" definitions for non-scientific concepts are extracted definitions, in which concepts are assigned meanings according to their use.

We highlight that, according to Tall and Vinner (1981), the conceptual definition consists of the symbolic way of specifying a concept. The authors distinguish between the formal conceptual definition, which is precise, and the personal conceptual definition, which is the personal understanding of a person's formal definition. To the authors, a formal definition in mathematics is a symbolic construction accepted by a large part of the mathematical community and is used to mean something specific. The personal conceptual definition, understood as a personal construction of the formal definition, refers to the conceptual image and, as it is personal, it may differ from the formal definition, i.e., the personal definition may be compatible or may differ from the formal definition.

According to Tall and Vinner (1981), we understand that the conceptual definition is related to the two definitions presented by Edwards and Ward (2008) since we understand that the conceptual definition can be stipulated or extracted. Formal mathematical definitions have stipulated meanings for the concepts they refer to. When the meanings of a mathematical concept are evoked from a formal definition, they are specific to the concept, and their use refers to this specificity; we understand them as stipulated definitions. The stipulated definition conveys an elementary meaning, guides a specific discussion, and is used to serve a purpose. The stipulated definition gives rise to the uses of concepts, while the extracted one arises from the uses and concepts.

2.3. Symbolic Interactionism

One of the main factors in the teaching and learning processes in the classroom is the interrelationship between teachers and students and between students and students, as it influences learning and interferes with the dynamics of relationships. According to Godino and Llinares (2000), a significant portion of research in mathematics education is dedicated to studying the relationships between teachers and students during classes when carrying out mathematical activities. In this section, we summarize the main characteristics of symbolic interactionism and its position in relation to learning, the notion of meaning, the role of the subject as a social being, and the interpretation of meanings.

In our study, symbolic interactionism was the theoretical basis for investigating and understanding how we perceive students' use of mathematical definitions in classroom discussions and group work presentations. In the context of the analysis, we focus on the relationships between teachers and students and between students and students, based on the notions of conceptual image and conceptual definition by Tall and Vinner (1981) and Vinner (1991) and definitions stipulated and extracted following Edwards and Ward (2008).

According to Blumer (1980, p. 119), symbolic interactionism is based on three premises:

> The first establishes that human beings act in relation to the world based on the meanings it offers them. [...] The second premise is that the meanings of such elements come from or are

> provoked by social interactions with other people. The third premise states that such meanings are manipulated by an interpretative process (and modified by it) used by the person when relating to the elements they come into contact with.

Blumer's propositions show that individuals interpret events and behave toward other individuals or objects based on the meanings they attribute to them. In other words, rather than simply responding to each other's actions, individuals interact with each other through mutual interpretation of actions. Individuals interpret the world around them interactively, and this social interaction is continuous and mediated by symbols and meanings.

Blumer (1980, p. 121) argues that "meaning is produced from the process of human interaction,"i.e., it is the result of interaction processes that originate or are provoked by social interaction and can undergo changes over time. The author also affirms that "symbolic interactionism considers meanings social products, creations elaborated in and through human activities that determine their interactive process" (Blumer, 1980, p. 121).

3. Methodological Path

This work is part of a broader dissertation (LOPES, 2014) developed in a classroom environment involving students in the first period of the information systems course at the State University of Montes Claros. The classes had six hours/classes per week, with two daily classes lasting 50 minutes each. During the period, eight activities were conducted covering functions and their derivatives, complemented by a seminar that presented the results derived from the students' studies.

The classes conducted by the professor in charge of the IDC subject took place in the teaching environment; the additional tasks proposed by the researcher were carried out in the computer laboratory (six tasks) and the teaching environment (two tasks). During the seminar, the students, who had previously grouped, conveyed their conclusions about research on functions and their derivatives, which were the focus of analysis in this study.

The professor taught the classes, formally covering previous knowledge (such as functions, variations of functions, methods of representing functions, trigonometric, rational, algebraic, logarithmic, exponential, and polynomial functions, among others) and the basic principles of limits and derivatives, following the approach outlined in the Calculus book used by the institution.

One of the leading professors' main concerns was to begin each explanation with exact and succinct definitions of the subject to be covered. For this purpose, he wrote the definition on the board and illustrated it with cases from the textbook. At the end of the class, the professor asked students to practice the exercises and challenges proposed in the book at home, making corrections in the subsequent class.

After completing some tasks, the professor and the researcher suggested holding a seminar in which the students would present, in teams, the results of their research on functions and their derivatives. This exposure approach would be the starting point of the evaluation of the application of mathematical definitions in the representations created by students throughout the activities.

Each team was responsible for exploring a theme associated with a function and its characteristics, examining the connections between that function and its derivatives. Collectively, students needed to summarize the results first, then conduct experiments using GeoGebra, and finish by presenting the conclusions of their research on functions and their derivatives in a seminar. These studies began in the classroom and the laboratory, with the guidance of the researcher and the professor. However, to complete the work, the students conducted research outside of class hours, during which the professor and the researcher did not guide them or intervene. The professors' observations were made through interactions with students, questions, and interventions when requested.

The activity proposed by the professor and researcher aimed to encourage and inspire students to conduct investigations and research based on the concepts discussed in practical activities and the definitions explored by the professors during classes. The intention was not for students to conduct a presentation on the chosen topic but rather to share with their colleagues the discoveries or conclusions reached about the type of function chosen and its derivatives. We emphasize that each group selected a function to be studied and researched according to its specific interest. In this article, we will exclusively address the data produced by the group that focused on studying exponential functions.

The research used instruments that made it possible to obtain detailed data. For this, resources such as the researcher's field notes, a virtual discussion platform, students' documents, and conversations captured in audio and video recordings were used. These records documented students' interactions with each other and between students and the professors, both during the execution of the eight proposed activities and in classes taught by the professors (the Professor and the Researcher). Although this information was not incorporated into the analysis, it formed the basis for interpreting the results.

We realized that the interactions between students, the professor, and the researcher played a significant role, highlighting some difficulties faced by students in using mathematical definitions while studying exponential functions and their derivatives. This scenario motivated us to conduct a more detailed investigation into understanding the mathematical definitions present in the students' mathematical expressions regarding these studies. In this way, we directed the data analysis to the transcriptions of the discussions in the seminar promoted by the groups of students.

The episodes selected for analysis were interpreted based on classification and categorization procedures (Charmaz, 2009) and qualitative content analysis (Graneheim; Lundman, 2004). At first, the data were classified systematically, and subsequently, the codes were grouped without reference to theoretical contributions. The contributions were later used as a perspective to interpret the grouping results, aiming at the research objective.

Therefore, we emphasize the systematic categorization, understanding, and representation of data as fundamental steps of this analysis method, consisting of a set of qualitative techniques that aim to find meaning in the data. Content analysis can be used for quantitative and qualitative approaches when applied in contexts with various data requiring interpretation.

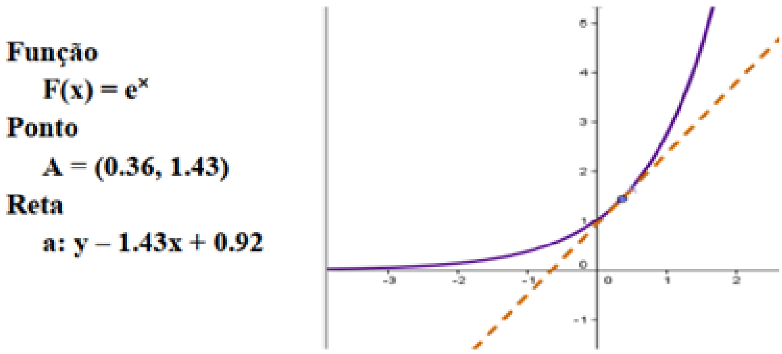
The data contained in the seminar recordings was organized first through coding, which required careful and concentrated reading of the data in search of ideas for analysis. We separated the data according to their scenario to create codes and organized them to express actions; that is, we coded the data as processes, using the nominal form of the verb in the gerund. According to Charmaz (2009), using gerunds in coding helps the researcher detect and focus on the processes present in the data, conveying a solid sense of movement and sequence.

4. Analyses and Discussions

Six students participated in this group, formed by two previous groups. They presented the work in two parts: Alice, Miguel, and Fábio presented the first, and Elen, Bela, and Márcio presented the second. To preserve anonymity, all students received aliases. The groups carried out studies separately and organized themselves into a single presentation. The studies focused on the behavior of the function derived from exponential functions to find maximum and minimum points, growth and decrease, and inflection points. At the beginning of the presentation, the students showed and explained the content of the slides: Slide 1—Definition of Exponential Function; Slide 2—Domain and Image of Exponential Function; Slide 3—Absolute Maximum; Slide 4—Local Maxima and Minima.

After reading each slide, the team presented the function graph in GeoGebra, aiming to show that the exponential function has no maximum or minimum point. Using the slider, they slid a tangent line along the curve to show that the function does not touch the x-axis, as shown in Figure 1.

Figure 1: Graph of function and tangent line



Source: Research data

Alice: *The polynomial team proved that this actually happens, but it doesn't happen in the exponential, because the exponential has no roots. [The student was talking about the work presented by the team that presented polynomial functions, which showed maximum and minimum points in the polynomial function].*

Miguel: *We tried everything; we researched the theory to be able to understand this here. It was a law, it was defined, so we did everything we could to find it. [Find a theory that showed a maximum and minimum point, and/or an inflection point specifically in the exponential function].*

Alice: *We also plotted in GeoGebra to see what the second derivative said. Whether the same thing happened in the polynomials.*

Miguel: *The first derivative had to be zero to achieve a maximum or minimum.*

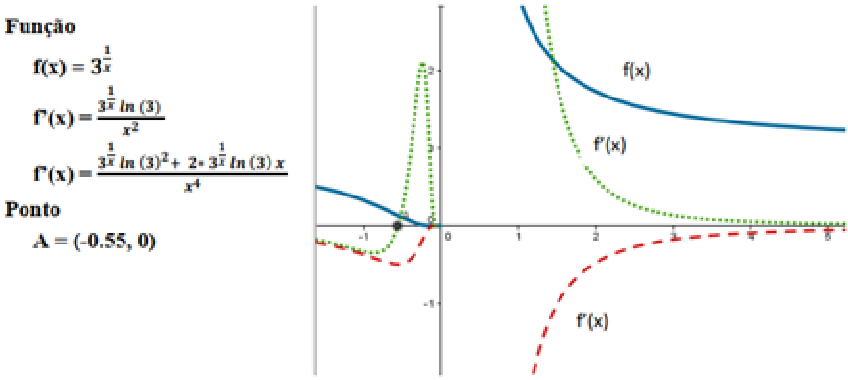
Alice: *We plotted this graph there, moved the function several times, and observed that the derivative does not change. [The derivative does not change because the derivative of the function is $f(x) = e^x$ and $f'(x) = e^x$].*

> **Miguel**: *We did the same thing as the other teams, we added and subtracted constants. The values change, but the derivative does not alter. No derivatives reach zero, they don't touch the x-axis. We zoomed in and got a lot closer, but it didn't touch the axis. We put a tangent line to try to prove that it doesn't touch it.*
>
> **Miguel**: *Since we were unable to find a derivative that touches the x-axis, we concluded that the exponential function has no maximum or minimum point.*

Miguel concluded that if the derivative does not touch the x-axis, the function does not have a maximum or minimum point. After this conclusion, the other part of the team presented the results of their studies about inflection points, which were added after defining the original objective of the framework. They started with the following question to the class: "Is it possible to have an inflection point in an exponential function?"

At that moment, most of the class said it was impossible. Then, Márcio presented the function in GeoGebra $f(x) = 3^{1/x}$ as an example of an exponential function and showed that even though the function did not have a maximum or minimum point, it had an inflection point, as shown in Figure 2.

Figure 2: Graph of function $f(x) = 3^{1/x}$, $f'(x)$, $f''(x)$ and point A



Source: Research data

> **Márcio**: *Take a good look at this function here: 3 to the power of one over x. When we took the second derivative of it, we got a root here, approximately -0.55 [see point A on the graph]. This proves that an exponential function has an inflection point, even though it does not have a maximum or minimum point.*
>
> **Bela**: *In all we researched, we did not find an exponential function with an inflection point. But we found this in one of James Stewart's exercises, which shows that there is; we have the references.*

Bela referred to example 8 in James Stewart's Calculus Book (2010, p. 274): Example 8: Sketch your graph using the first and second derivatives of $f(x) = e^{1/x}$, along with the asymptotes

When asked why they did not use that example, they said that the exponential function had to have a number base, and the number e was a letter, not a number, so they changed the letter e by the number 3 in the function. Because the members of this group had carried out studies separately, this justification was given by a student who did not participate in the presentation of the graph of function $f(x) = e^{1/x}$ in Figure 1; therefore, it went unnoticed. The students started arguing, and his classmates convinced him that the number e represented Euler's number, approximately 2.71828, so it could have been used as a basis for exponential functions. After some questions, they started debating about Euler's number and why is an exponential function, so the professor resumed the discussion with the following question:

> **Professor**: *What exactly is the function?*
>
> **Márcio**: *$f(x) = 3^{1/x}$.*
>
> **Researcher**: *Before James Stewart, you thought there was no inflection point, is that it?*
>
> **Márcio**: *Until this morning, we believed that this function did not exist.*
>
> **Bela**: *In fact, it's the following: We found an article from a federal university in Rio de Janeiro that we thought was safe. We saw a function with an inflection point, and we were unsure whether it was exponential.*
>
> **Researcher**: *How many times have you changed your opinion?*
>
> **Bela**: *We changed [our opinion] ten times or more. But then, this morning, after we saw this in James Stewart's book, the doubts ended, because his book is reliable; it's the teacher's book.*
>
> **Guto**: *Sorry to ask now, but is this function exponential?*
>
> **Márcio**: *This one is; $f(x) = 3^{1/x}$ is exponential.*
>
> **Marcelo**: *Isn't it irrational?*
>
> **Researcher**: *Is it exponential, rational or irrational?*
>
> **Márcio**: *The criterion for an exponential function is that its base must be raised to an exponent x. Here, we have base 3 and exponent 1/x. The exponent is considered as a variable, so it is an exponential function.*

Most students were engaged in discussions about function $f(x) = 3^{1/x}$. The group that carried out studies on irrational functions said that the function was irrational, and the group that studied polynomials said it was a polynomial. They resorted to definitions and defended their point of view, and even when questioned by the professor, they maintained that the function presented characteristics of an exponential. Based on the definition of the exponential function presented at the beginning of the work exposition (be $a \in R, a > 0$ e $a \neq 1$, we call an exponential function the function defined by $f(x)= a^x$), the student was convinced that function $f(x) = 3^{1/x}$ was an exponential.

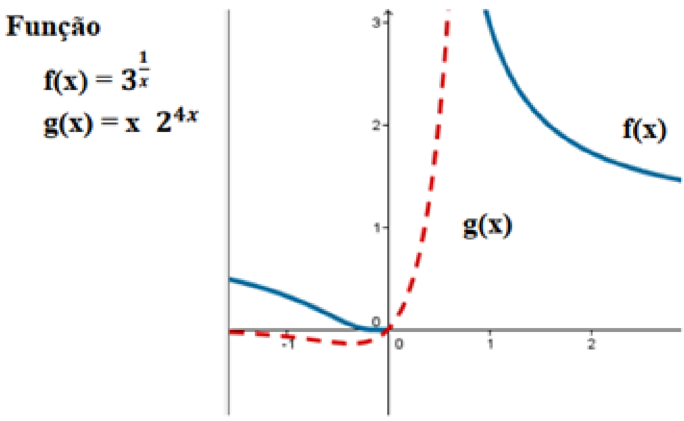
This episode takes us back to Vinner's (1991) conclusions regarding the role of definition in mathematics. This author considers there is a severe problem in learning mathematics around definition apprehension, especially regarding the conflict between the mathematics structure and the cognitive processes of acquiring mathematical concepts. When Márcio states: "The criterion for having an exponential function is to have a base raised to an exponent x", his ground was the formal definition of an exponential function.

He observes the base 3 and exponent 1/x, and concludes that the function is exponential. In this case, he did not take into account that $f(x) = a^x$ is different from $f(x) = a^{1/x}$. In other words, having the variable in the exponent is not equivalent to having the variable as an exponent. The student knew and stated the formal definition of an exponential function and even used it to support his conclusions. However, he did not understand the formal mathematical meaning of $f(x) = a^x$. In this case, we can see that just knowing the formal definition does not guarantee the student's understanding. Vinner (1991, p. 6) also affirms this idea: "We assume that acquiring a concept means forming a conceptual image for it. Knowing the conceptual definition of color does not guarantee understanding the concept."

> **Professor**: *Will the team keep the idea that it is exponential?*
>
> **Bela**: *We did all the work stating that it is.*
>
> **Professor**: *So, looking at the definition like this, you still think it is exponential, is that it?*
>
> **Márcio**: *Yes.*
>
> **Researcher**: *So let's take a look in GeoGebra. If you plot its graph, will it be exponential?*
>
> **Bela**: *I see the graph of an exponential, I do.*

The student returned to GeoGebra and plotted the graph of the following functions:

Figure 3: Graph of function f and function g



Source: Research data

When analyzing the two graphs, the following discussion ensued:

> **Researcher**: *What does the graph of an exponential function look like? Go back to the graph. There's also something I don't understand, Márcio. Why do you claim that* $f(x) = 3^{1/x}$*. is exponential and* $f(x) = x.2^{4x}$*. is not? What's the difference?*
>
> **Márcio**: *Our analysis was according to the definition: the first has a base and the variable is in the exponent. Yet for the second, the variable is in the exponent, but it is also in the base.*
>
> **Professor**: *Ok, but look graphically: does it look like an exponential function?*
>
> **Márcio**: *I can see an exponential function, I do. Look at the curve here; it's identical. [The student mentioned the graph of function* $f(x) = 3^{1/x}$ *represented only in the first quadrant].*
>
> **Researcher**: *And what about this part that is in the second quadrant?*
>
> **Márcio**: *This here? I discarded it, just like that. Is this part of the graph? I didn't know.*

The professor retook the floor, re-read the definition of an exponential function with the class, and showed the characteristics of the functions on a graph. The interactions between group members and teachers and the arguments used in the discussions lead us to Vinner's (1991) statements. The author developed a model based on two cells, one for the conceptual image and the other for the conceptual definition and considers them central to explaining the cognitive process of concept formation.

Vinner (1991, p. 11) considers that "no matter how your association system reacts when a problem is placed in a technical context, you are not expected to formulate your solution before consulting the conceptual definition." This is, naturally, the "desirable process"; however, the author recognizes that this does not correspond to what the student performs in practice. Tall and Vinner (1981) state that the conceptual definition, generally used to develop mathematical concepts in university teaching, comprises the formal and personal conceptual definitions.

In our study, we can infer that Márcio's conceptual image cell of the exponential function was gradually given new meaning based on his professors' and classmates' examples and arguments. According to Vinner (1991), many teachers expect a one-way process for concept formation. They expect the conceptual image to be formed through the formal conceptual definition and to be completely controlled by it.

Clearly, we realize that the student's conclusions and their answers to questions regarding the exponential function are not coherent with the formal conceptual definition. However, we realize that throughout the process, the personal conceptual definition was constructed from the formal conceptual definition of the exponential function, as Márcio says: "We analyzed according to the definition; the first has a base and the variable is in the exponent." The meanings of this mathematical concept were evoked from the formal conceptual definition, and, thus, we have a stipulated definition, as its personal definition was directly based on the formal definition, however, without understanding the specfic meaning stipulated to the exponent x.

Arguments based on his conceptual image regarding what an exponential function is have emerged from various contexts of use, coherently or not, with the specificity of the formal definition. Thus, Márcio, without understanding the meaning stipulated for the exponent x for the concept of exponential function, attributed meanings to this concept.

After the professor's observation, José made a pertinent comment:

> **Professor***: See, we broke down your arguments one by one, and James Stewart didn't say it was exponential, right?*
>
> **Bela***: No, we've concluded this. It seemed to be.*
>
> **José***: Do you agree that the exponential function has only one concavity? How will it have an inflection point if it doesn't change concavity? Where is the logic of that? If it only has one concavity, how could that be?*
>
> **Researcher***: And now? What can you say?*
>
> **Márcio***: We now state that these functions that we have placed here are not exponential and exponentials have no inflection point.*

Finally, we observed that Márcio's last statement reframed his personal conceptual definition, that is, in aspects of his extracted definition. When he says: "We now affirm that these functions that we put here are not exponential and exponentials do not have an inflection point," we realize that he attributed meanings to the concepts discussed, referenced by their definitions, according to their use in this situation.

José used concepts from the study of functions and their derivatives in his arguments. Vinner (1991) recommends, when necessary, initiating cognitive conflicts with students to encourage them to a higher intellectual stage and states that one of the goals of teaching mathematics should be to change habits of thought from the routine mode to the technical mode.

Thus, if their nature allows, mathematical concepts should be acquired in the routine mode of concept formation and not in the technical mode of doing it. One should begin with several examples and counterexamples through which the conceptual image will be formed. For him, definitions can play critical roles in technical contexts, and these impose on the student some habits of thought that are entirely different from those typical of everyday life. At the beginning of the learning process, everyday thinking habits override the thinking habits imposed by technical contexts, and students also continue to use routine thinking habits in technical contexts.

5. Final Considerations

This research was motivated by concerns arising from a personal experience as a Calculus professor related to the difficulties frequently faced by university students while learning the topics covered in the context of the IDC. The specialized literature in mathematics education shows that Calculus has played a prominent role in investigations, both because it is one of the main causes of student failure and because of its prominent position in the configuration of advanced thinking in mathematics (Igliori, 2009).

We noticed that the dynamics between the students, the professor, and the researcher during the execution of the tasks and the presentation of the results by the groups were significant, highlighting some of the students' deficiencies in terms of applying mathematical concepts and developing a consistent argument to support the propositions identified during the execution of the activities, which addressed the analysis of functions and their respective derivatives.

When examining the events chosen in our research, we concluded that interaction constitutes a social procedure. Although students have different conceptual conceptions individually, these ideas are generated collectively, and can be changed by the student throughout the interactions that occur in a specific context, as exemplified in the seminar promoted for the presentation, discussion, and formalization of concepts they associate with different functions and their derivatives.

The group's studies focused on the behavior of functions derived from exponential functions to find maximum and minimum points, growth and decrease, and inflection point. Among the episodes analyzed from this group's presentation, we highlight that the topic lecture in the seminar began with formal conceptual definitions, one of which conveyed the definition of an exponential function. Although the group projected this definition through the data show, the stipulated definition was not understood. Most students engaged in discussions about the function proposed by Márcio: .

Students used the formal conceptual definitions included in the work presented in the seminar and supported their point of view. Despite the professor's questions, they maintained the conviction that the function had characteristics of an exponential function. This incident leads us to Vinner's (1991) conclusions about the role of definitions in mathematics. This researcher observes a significant challenge in mathematics learning related to understanding definitions, mainly regarding the conflict between the mathematics structure and the cognitive processes of assimilating mathematical concepts.

When Márcio states: "The criterion for having an exponential function is to have a base raised to an exponent x," his ground was the formal definition of an exponential function. He observes the base 3 and exponent , and concludes that the function is exponential. In this case, Márcio did not take into account that is different from . The student knew and stated the formal conceptual definition of the exponential function and used it to support his conclusions; however, he did not understand the formal mathematical meaning of .

After the interactions during the seminar, we observed that Márcio's last statement reframed his personal conceptual definition, that is, aspects of his extracted definition. When he says: "We now affirm that these functions that we put here are not exponential and exponentials do not have an inflection point," we realize that he attributed meanings to the concepts discussed, referenced by their definitions, according to their use in this situation. We corroborate Vinner (1991, p. 6), when he states that: "We assume that acquiring a concept means forming a conceptual image for it. Knowing the conceptual definition of color does not guarantee understanding the concept." Thus, the fact that the student knows the formal conceptual definition does not guarantee that he has reached an understanding of it in the sense of the stipulated definition.

The results of our research suggested that students' understanding of functions and their derivatives progressed through interactions between them, the class professor, and the researcher.

This article aims to contribute to the development of pedagogical practice by introducing fundamental IDC concepts. It refers to a particular domain: the conceptual image and conceptual definition of exponential functions and their derivatives. We hope this article contributes to the apprehension and debate on the study of functions, exponential or not, and their derivatives. We also hope it can represent a contribution to other mathematics educators and, mainly, to professors who work in mathematics teaching degrees, as well as subsidizing other research and motivating other researchers.

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Editorial History
Received on 07/01/2024.
Accepted on 10/04/2024.
Published on 18/07/2024.

How to cite – ABNT

OLIVEIRA, Saulo Macedo de; LOPES, Rieuse. Exponential Functions and their Derivatives in the light of Conceptual Image and Conceptual Definition. **REVEMOP**, Ouro Preto/MG, Brasil, v. 6, e2024007, 2024. https://doi.org/10.33532/revemop.e2024007

How to cite – APA

Oliveira, S. M. de, & Lopes, R. (2024). Exponential Functions and their Derivatives in the light of Conceptual Image and Conceptual Definitio. *REVEMOP*, *6*, e2024007. https://doi.org/10.33532/revemop.e2024007

This article was funded by the Minas Gerais State Research Support Foundation (Fundação de Amparo à Pesquisa do Estado de Minas Gerais–FAPEMIG), Project APQ-04960-23, Notice N. 008/2023–Support program for scientific and technological publications.