Articles Section
2024, Volume 6, e2024009
Pages 1–18
10.33532/revemop.e2024009

revemop
ISSN 2596-0245

Teaching Mathematics through Problem Posing: four practices for handling students' posed problems

Ensino de Matemática por meio da Proposição de Problemas: quatro práticas para lidar com problemas propostos pelos alunos

Enseñar matemáticas por medio la Proposición del Problema: cuatro prácticas para abordar los problemas propuestos por los alumnos

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Abstract

Problem posing is both a learning goal and an effective instructional approach, but what are the details of using this approach to foster students' learning? In this paper, we use teaching cases to demonstrate a step-by-step instructional model that teachers can use to convert non-problem-posing tasks into problem-posing ones and use the tasks to foster students' learning. More importantly, we provide a detailed illustration of the four practices that teachers and students can use to handle students' posed problems. The paper ends with a discussion of future work to validate and improve the model and the challenges that teachers experience teaching through problem posing.
Keywords: Problem Posing; Teaching Mathematics Through Problem Posing; P-PBL Instructional Model; Student Learning; Teaching Practice.

Resumo

A proposição de problemas é tanto uma meta de aprendizagem, quanto uma abordagem instrucional eficaz, mas quais são os detalhes do uso dessa abordagem para promover a aprendizagem dos alunos? Nesse artigo, usamos casos de ensino para demonstrar o passo a passo de um modelo instrucional que os professores podem usar para converter tarefas que não são de proposição de problemas em tarefas de proposição de problemas e usá-las para promover a aprendizagem dos alunos. Mais importante ainda, fornecemos uma ilustração detalhada das quatro práticas que professores e alunos podem usar para lidar com os problemas propostos pelos alunos. O artigo termina com uma discussão do trabalho futuro para validar e aprimorar o modelo e os desafios que os professores enfrentam ao ensinar através da proposição de problemas.
Palavras-chave: Proposição de Problemas; Ensino de Matemática Através da Proposição de Problemas; Modelo Instrucional P-PBL; Aprendizagem de Aluno; Prática de Ensino.

Resumen

Plantear problemas es tanto un objetivo de aprendizaje como un enfoque didáctico eficaz, pero ¿cuáles son los detalles de la utilización de este enfoque para promover el aprendizaje de los alumnos? En este artículo, utilizamos casos de enseñanza para demostrar un modelo de instrucción paso a paso que los profesores pueden utilizar para convertir tareas que no son de resolución de problemas en tareas de resolución de problemas y utilizarlas para promover el aprendizaje de los alumnos. Y lo que es más importante, ofrecemos una ilustración detallada de las cuatro prácticas que profesores y alumnos pueden utilizar para tratar los problemas propuestos por los estudiantes. El artículo termina con una discusión sobre el trabajo futuro para validar y mejorar el modelo y los retos a los que se enfrentan los profesores cuando enseñan a través del planteamiento de problemas.
Palabras clave: Resolución de Problemas; Enseñanza de las Matemáticas a través de la Resolución de Problemas; Modelo Instruccional P-PBL; Aprendizaje de los Alumnos; Práctica Docente.

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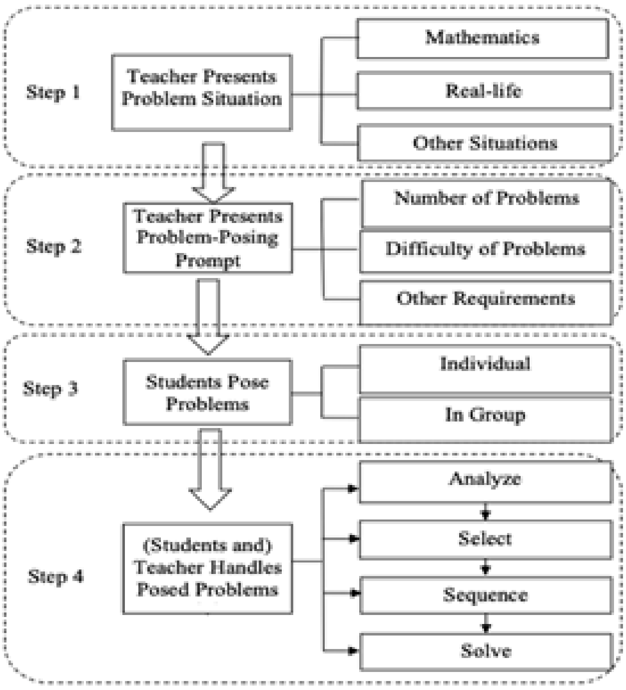
1. Introduction

Cai *et al.* (2015) raised 10 active questions in problem-posing research, one of which was: What does a classroom look like when students engage in problem-posing activities? Although the field has made significant progress toward understanding some elements of classroom instruction that uses problem posing, such as how to design problem-posing tasks, little effort has yet been made to understand how to handle students' posed problems. In particular, teachers face challenges when they must handle, within a limited timeframe, the numerous problems students pose--problems that can have very different cognitive levels. For example, given the need to address the teaching goal, teachers are sometimes unsure if they should even use their students' posed problems at all or if they should simply use the problems given in the textbook. When they do choose to use their students' posed problems, they may be unsure about which problems to select for the whole class to attend to. Moreover, teachers may not have effective strategies for dealing with posed problems that are nonmathematical. In this paper, we intend to provide an instructional model with specific steps and practices that can address these issues.

This paper includes four sections. In the first section, we focus on why problem posing is important in mathematics education. In the second section, we review research on how teachers implement problem posing as an instructional approach in their classes and propose an instructional model (Problem-Posing-Based Learning [P-PBL]; Cai, 2022; see Figure 1). In the third section, we use examples to illustrate and examine this instructional model with a focus on the four practices in Step 4 of the model (analyze, select, sequence, and solve), and we provide some guidance to teachers. In the last section, we discuss the necessity of handling students' posed problems and the associated challenges for teachers.

Figure 1 - Instructional model P-PBL



Source: Cai (2022).

2. Importance of problem posing

Problem posing is seen as an important activity in mathematics education globally, as indicated by the curriculum standards in countries such as Australia, China, the United States, and Turkey, all of which position problem-posing tasks as an important component of their curriculum (Arikan; Unal, 2014). Also, problem posing is beneficial to both teachers and students. It has potential to enhance learners' cognitive and affective outcomes while also offering teachers some pedagogical advantages. In this context, cognitive outcomes refer to students' domain-specific knowledge related to mathematical facts, concepts, and procedures (Klein *et al.*, 2005), and students' mathematical achievement. Affective outcomes, as Schindler and Bakker (2020) suggested, include the following constructs: emotions, attitudes, beliefs, self-efficacy, interest, motivation, and values. Pedagogical advantages (Li *et al.*, 2020) pertain to the advantages of using effective teaching strategies and methods to enhance learning outcomes. We explain each of these three aspects in detail below.

2.1. Learners' cognitive outcomes

Problem posing can improve students' cognitive outcomes. Cai and Hwang (2023) suggested that although problem-posing tasks are cognitively demanding, they can be adapted to different students' abilities. That is to say, problem posing can not only challenge mathematically mature students' mathematical thinking but also stimulate the mathematical thinking of students with lower mathematical achievements. This is supported by Silber and Cai's (2021) finding that underprepared undergraduates were capable of posing mathematical problems. Moreover, problem posing can adapt to learners at different learning stages from elementary to higher education. Toluk-Uçar's (2009) study on the effect of problem posing on preservice teachers' understanding of fractions showed that these teachers had a better conceptual understanding of fractions when they were taught through problem posing. A study of elementary students showed that teaching through problem posing significantly improved students' problem-solving skills (Kopparla *et al.*, 2019). Studies have also shown that problem posing can promote students' understanding of mathematics knowledge and improve students' performance on mathematical achievement tests. For example, Ozdemir and Sahal's (2018) study showed that teaching through problem posing had a positive influence on students' mathematical achievement.

2.2. Learners' affective outcomes

Problem posing also has a positive influence on students' affective outcomes (Cai; Leikin, 2020). Problem posing can position students as agents (National Council of Teachers of Mathematics [NCTM], 2020) and simulate students' interest in mathematics learning (Silver; Cai, 1996). Schindler and Bakker's (2020) case study of a student's affective outcomes showed that the student's attitude toward mathematics changed from viewing mathematics as boring to viewing mathematics as fun, and her perceived value of mathematics changed from emphasizing the outcomes to accepting being wrong when involved in an "extracurricular, inquiry-oriented collaborative problem posing and problem solving program" (p. 303). Hendriana *et al.* (2019) showed that 11th-grade students' learning was more active in the problem-posing approach class compared to the learning in the conventional learning classroom and students showed positive opinions toward the implementation of the problem-posing approach in the classroom. Studies have also shown that students' interest in learning mathematics can be enhanced in a problem-posing teaching context (Walkington; Bernacki, 2015).

2.3. Pedagogical Advantages

Problem posing not only benefits students but also teachers. Ball and Forzani (2009) suggested that problem posing provided teachers with opportunities to access and understand students' thinking. Li *et al.* 's (2020) study on teachers' beliefs regarding problem posing also suggested that teaching through problem posing provided teachers with opportunities to gain a better understanding of their students' mathematical thinking. Meanwhile, problem posing enables teachers to provide students with more learning opportunities and to accommodate individual differences. However, only a comparatively small number of teachers in Li *et al.*'s study recognized these pedagogical advantages. Lin's (2004) study also showed these pedagogical advantages but without providing the percentage of teachers who did so.

These results indicate that there is still great room for exploration in the pedagogical advantages area. The insufficient exploration might be caused by (a) current research has primarily focused on students' learning, both the cognitive and affective outcomes of problem posing, instead of the teachers' teaching, one of which is related to how teachers can benefit from teach mathematics through problem posing; (b) a lack of problem-posing tasks in the existing curriculum (Cai; Hwang, 2021), indicating that there might be insufficient guidance to teachers to teach through problem posing. To empower teachers to reinterpret and reshape existing curriculum materials (Cai; Hwang, 2021) as well as to handle students' posed problems, we next describe what teaching through problem posing looks like with a focus on what a problem-posing task looks like and how to handle students' posed problems.

3. What teaching mathematics through problem posing looks like

Teaching through problem posing refers to teaching that includes problem-posing tasks for students. Problem-posing tasks are instructional tasks that involve problem-posing situations and prompts that initiate students' problem-posing activities (Cai; Hwang, 2021). To support teachers' teaching through problem posing, Cai (2022) proposed the P-PBL instructional model (Figure 1). This model demonstrates the routine for one problem-posing task. Below, we use several examples to illustrate the four steps in this model. Steps 1 and 2 present the components of a problem-posing task, which are useful for teachers to create a problem-posing task or reshape a pre-existing problem-solving task into a problem-posing task. Step 3 provides guidance to teachers on how to organize the class for problem-posing activities, and Step 4 provides detailed practices for handling posed problems.

3.1. Step 1: teacher presents problem situation

The first component of a problem-posing task is the problem situation. In Step 1, teachers present a problem situation for students to pose mathematical problems. We categorize problem situations into three types according to prior studies: real-life, mathematical, and other situations. Real-life situations are about dealing with the natural situations that happen in their everyday life (Bonotto, 2013). For example, teachers presented students with supermarket and travel expense situations (Bonotto, 2013) and milk-drinking, apple, and pizza situations (Toluk-Uçar, 2009). Another typical type of the problem situation is the mathematical situation, which includes mathematical expressions, graphs, concepts, or properties. An example of a mathematical situation would be mathematical expressions (e.g., 5 x 8, 120 - 30) as used in Cai and Hwang (2022). Other situations refer to situations that are neither real-life situations nor mathematical situations. In Example 1 below, the teacher presented students with a real-life situation.

Example 1: Volume and Units of Volume

In this problem-posing task, the teacher showed two videos. The first video showed two cups of the same size, No. 1 and No. 2, with the same amount of water. The teacher added stones to cup No. 1. As more stones were added to the cup, the water surface rose slowly (Figure 2). In the second video, the No. 2 cup was filled with sand and poured into the No. 1 cup of the same size. There were plastic cuboids in cup No. 1, so one third of the sand remained in cup No. 2 (Figure 3).

No. 1 No. 2

Figure 2: cups with water

Source: Researchers' archive (2022)

No. 1 No. 2

Figure 3: cups with sand

Source: Researchers' archive (2022)

Teacher: What mathematical problems can you pose based on these two videos?

Student 1: Rocks take up space for water, and plastic blocks take up space for sand. Do all objects take up space?

Student 2: Some rocks are big. When they get into the bottle, the water rises more, and the small ones rise less. Does it mean how much space each object occupies?

Student 3: Is the size of the space occupied by an object the same as its volume?

(Dialogue between teacher and student, 2022)

3.1.1. Guidelines for teacher to presente of problem situation

a) A teacher might begin with a discussion about a related real-life topic before showing students information from a specific situation (e.g., talking about climate change, sea level rise, and possible impacts to coastal communities before showing the students data on temperatures or sea levels over the past few decades).

b) If a teacher plans to use a relatively complex mathematical situation for students to pose problems, they might present some simpler and relevant mathematical situations before they show the targeted one. The reason is that students can be clearer about what problems they can pose. For example, in Cai and Hwang's (2022) study, before asking students to pose problems according to the mathematical situation 1200-3x=? The teacher first asked students to pose problems according to 5x8, and 120-30 respectively to help students see how each expression matches to different contexts. By making sense of the corresponding contexts to multiplication and those to subtraction separately, students are likely to pose mathematical problems according to the more complex mathematical situation 1200-3x=?.

3.2. Step 2: teacher presents problema-posing prompts

The second component of a problem-posing task consists of the problem-posing prompt. Cai (2022) classified prompts by number of problems, difficulty of problems, and other requirements. Regarding the number of problems, Stoyanova and Ellerton (1996) used prompts like "please pose as many problems as you can" and "ask as many questions as you can." Silver and Cai (1996) used the following prompt: "Write three different questions that can be answered from the information below." Regarding the difficulty of problems, Silber and Cai (2017) used the prompt "write one easy problem, one moderately difficult problem, and one difficult problem using the information in the situation" (p. 166) and the prompt "please write challenging mathematics problems that you would be able to solve" (p. 169). Other requirements might consist of asking students to pose an interesting problem (Baumanns, 2022) or a related problem (El Sayed, 2002).

3.2.1. Guidelines for teacher to presente of problem-posing prompts

a) If it is difficult for students to pose some problems based on the problem situation using a prompt like pose three mathematical problems, especially when the students are not familiar with problem posing activities, the teacher can show one or two sample posed problems for the problem-posing situation and then use the prompt pose similar problems for students to pose problems. Posing similar problems might be easier for students because they will have a better sense of what they should do. However, it might limit students' thoughts sometimes. Therefore, we also suggest teachers try prompts without using sample questions when students show familiarity on problem posing activities.

b) Based on our conversations with teachers who have tried teaching mathematics through problem posing, we suggest that problem-posing prompts be as specific as possible. Teachers can ask students to pose three problems instead of to pose "some" problems. Also, teachers can ask students to pose three *mathematical* problems rather than simply three problems. Once students have more experience posing problems, teachers might ask students to pose three mathematical problems that can challenge their classmates instead of posing any three mathematical problems.

c) There is no single "best" prompt. Teachers can try different problem-posing prompts with their students for different problem-posing situations to decide what works well for their own class.

3.3. Step 3: students pose problems

A problem situation and a problem-posing prompt together make a problem-posing task. Once a problem-posing task has been presented, teachers can organize their students to pose problems. In Step 3, we introduce two typical ways that teachers arrange students to interact with problem-posing tasks—in groups or individually (Chang *et al.*, 2012; Silber; Cai, 2017)—based on their problem-posing goals. In Silber and Cai (2017), participants were asked to write their posed problems on paper individually, and their responses to the problem-posing tasks as well as from individual interviews were used to better understand each student's understanding and thought process. Other studies have asked students to work in pairs (Bonotto, 2013) or in small groups (Kontorovich *et al.*, 2012) to pose problems. Group work serves purposes like providing chances for students to assert their ideas, receive feedback from peers, and generate innovative ideas. In Example 1, students posed problems individually. In a typical classroom, each student can contribute their talent and knowledge in posing problems individually as well as in group work. Teachers can adjust the ways they arrange students for problem posing based on their teaching goals.

3.3.1. Guidelines for teacher to organize the class to pose problems

a) When students are presented with a problem-posing task and work individually, it is possible that students cannot well understand the problem situation. As a result, they cannot pose high-quality problems, or even worse, they cannot pose problems without teachers' help. In this case, teachers should observe the class. If they find that only a few students cannot pose problems independently, they can offer help to these students by one-on-one guidance. However, if the number of students who cannot pose problems without help is great, and the teacher cannot offer one-on-one help to each of them during the expected problem posing activity time. We suggest teachers stop the individual problem posing activity and to start the whole class problem posing activity.

3.4. Step 4: (Students and) Teacher handle posed problems

In Step 4, both the students and the teacher handle the students' posed problems. Zhang and Cai's (2021) analysis of 22 teaching cases included teachers handling students' posed problems using several strategies. A typical pattern of handling posed problems included several steps. First, the teachers skipped the problems that were irrelevant to their teaching goals. Then, they classified the relevant problems based on difficulty level. The teachers mainly spent time on discussing the moderately challenging problems in class and asked the whole class to quickly solve the easier problems. The teachers assigned the very challenging problems to students for homework or left them for the next class as instructional goals. We summarize the strategies identified in this study as: analyzing posed problems according to their difficulty levels, selecting problems for whole-class teaching, and directing students to solve problems.

We have also observed some strategies in handling students' posed problems in other studies. For example, in Bonotto's (2013) study, the teacher asked students to work in pairs to pose problems, after which the students were expected to solve the problems of another group and the whole class discussed possible errors or inconsistencies in the posed problems. Toluk-Uçar's (2009) study used a similar strategy wherein students were asked to critique and modify the problems. The strategy we identified in these two studies is that teachers and students analyzed the posed problems together. Toluk-Uçar's study also explicitly showed that the students modified the problems after analyzing the problems.

These studies reveal certain similarities in the ways that teachers handle students' posed problems, such as jointly analyzing and identifying possible errors in the problems and having students solve them. From an improvement science (Bryk *et al.*, 2005) perspective, however, more specific and structured strategies can be developed to better support teachers to handle students' posed problems. We contend that teachers, particularly those unfamiliar with the problem-posing approach, would benefit from a set of practices to guide them in this regard. Cai's (2022) P-PBL Routine, specifically, Step 4 of the routine, can serve as a useful starting point for teachers seeking to employ problem posing in their teaching. In the next section, we delve into four practices that can aid teachers in handling students' posed problems.

4. Four practices for handling students' posed problems

This section outlines four important practices that can aid teachers in handling students' posed problems. These practices include (1) analyzing students' posed problems, (2) selecting some problems for students to discuss and solve, (3) sequencing the selected problems, and (4) solving problems. Although each of the practices has been discussed previously (e.g., Zhang; Cai, 2021), our contribution here is to synthesize that work and present an explicit and structured framework. Before presenting the details of the four practices, two more things we want to emphasize are that (a) there is no strict sequence for these four practices and (b) not all the practices have to be employed in a single problem-posing activity. The aim of the problem-posing activity, the nature of students' posed problems, and the students' reactions can influence teachers' implementation of the practices. For example, if all the problems are related to the teaching goal, it may be unnecessary for the teachers to select and discard some of them.

4.1. Practice: analyzing students' posed problems

Students' posed problems vary in different aspects, including focus, difficulty, clarity, and correctness. It would be better if teachers and students can analyze the posed problems to ensure that all of them are understandable and solvable as well as aligned with the teaching goal. Like Bonotto's (2013) study and Cankoy's (2014) study, the teacher asked the whole class to think about whether the posed problems were solvable. If a posed problem was not deemed solvable, the students could be asked to modify the problem. The practice of analyzing the problems can be carried out by teachers, students, or through collaborative effort. For example, teachers can provide feedback on students' posed problems and students can make modifications based on the feedback (Suarsana *et al.*, 2019). Here, we have two foci related to the practice of analyzing posed problems: classifying and modifying the posed problems. We present one example, similar to the studies mentioned here, of providing feedback and modifying problems along with an example of classifying problems while taking into consideration the time constraints of a typical classroom.

4.2. Analyzing: classifying the posed problems

Classifying the posed problems is putting the same types of problems together. Classifying posed problems is rarely mentioned in problem-posing research. However, we view classifying posed problems as an important practice for two reasons. First, dealing with a certain type of problem more than once can enhance students' understanding. Second, classifying the problems can make the best use of class time without spending too much time on the same type of problems, which will not contribute to students' understanding after several explanations and might decrease students' learning motivation by practicing the same type of problems. By classifying the problems, teachers can decide how much effort should be put into one type of problem based on students' reactions. This is an important instructional practice considering the constraints imposed by class time and class size, because it is hard to discuss every student's posed problems. By classifying the problems, teachers can save time and not feel rushed because they have to deal with too many problems. Moreover, most students will see that their problems or problems similar to theirs will be discussed by the whole class, which can help students feel valued and be more engaged.

In Example 2 below, the teacher presented all the posed problems to the students (see Figure 4) and asked if they had any time-saving ways to solve all the problems. In this example, the teacher guided the students to analyze the problems by classifying them into different categories. Student 1 said that they could solve one problem of each type. In other words, the students needed to classify the problems into types first and then select a representative of each type. Then, Student

2 classified Problems ② and ⑥ in the same category and explained the meaning of the category to be problems that ask for a percentage of a certain number.

Example 2: Percent Learning

Figure 4: The teacher presents the questions posed by each group on the blackboard.

Posing questions about percentage

①There are 40 girls in the music group, accounting for 50% of the whole group. How many people are in the whole group? 40 ÷ 50%=80 (person)	②There are 40 girls in the class, and the number of boys is 50% of the number of girls. How many boys are there? 40 x 50%=40 x 0.5=20 (person)	③There are 40 students in the fifth grade, and the number of boys is 50% more than the number of girls. How many boys are there? 40 x (1+50%)=40 x (1+0.5)=40 x1.5=60(person)
④Class three has 40 boys and the number of girls is 50% more than that of boys. How many people are there in total? 40 ÷ 50%+40=120 (person)	⑤There are 40 boxes of bananas in the supermarket, which is 50% more than the boxes of apples. How many boxes are there of apples? 40 ÷ (1+50%)=40 ÷ 150%=40 ÷ 1.5=60(box) ≈26	⑥There are 50 boys in the baseball team, and the number of girls is 40% of the number of boys. How many girls are there? 50 x 40%=50 x 2/5=20 (person)

Source: Researchers' archive (2019)

> *Teacher: With so many problems, solving them one by one is time consuming and labor intensive. What good solutions do you have?*
>
> *Student 1: Similar to what we did before, we only need to solve one problem of the same kind.*
>
> *Teacher: You mean, we first need to...*
>
> *Students: Classify.*
>
> *Teacher: Let's take a look. Which questions are of the same type? How can the questions be classified? If you have any difficulty, discuss with the group quietly.*
>
> *Student 2: I think ② and ⑥ are in the same category. They both ask for a percentage of a certain number.*
>
> *(Dialogue between teacher and student, 2019)*

4.3. Analyzing: modifying the posed problems

As discussed at the beginning of this section, and as most studies (e.g., Toluk-Uçar, 2009) have shown, many of students' posed problems need modification. Students' insufficient understanding, loose thinking, and language difficulties can result in unsolvable problems which should be clarified or discarded. Modifying the posed problems can help students better understand the problems and check their understanding of the mathematical concepts. In the meantime, students can become more engaged when they are modifying problems posed by their classmates. For Problem ⑤ in Example 2 (see Figure 5), the teacher raised the concern that something was wrong with the calculation: "There are 40 boxes of bananas in the market, which is 50% more than the boxes of apples. How many boxes are there of apples?" The teacher encouraged students to think about what they could do to fix the problem, and two students showed different ways of correcting the initial problem.

Why is there an issue with Problem ⑤? If we only focus on the quantitative relationship, Problem ⑤ is a solvable problem. However, further calculation showed that the result is not a whole number, which contradicts the background situation which requires that the number of boxes of apples be a whole number. To make Problem ⑤ solvable, students needed to modify this posed problem.

In this example, the teacher's concern was raised by the problematic answer rather than the posed problems. Teachers and students not only need to consider if the posed problems are understandable, but they also need to check whether the problems are solvable within the students' knowledge when the posed problems are understandable. In this example, the teacher worked as the mediator and helped students better understand the posed problems. There could be other occasions in which the teacher analyzes the problems individually.

Example 2 Continuation:

Figure 5: Posed Problem ⑤

⑤There are 40 boxes of bananas in the supermarket, which is 50% more than the boxes of apples. How many boxes are there of apples? 40 ÷ (1+50%) = 40 ÷ 150% = 40 ÷ 1.5=60(box) ≈26

Source: Researchers' archive (2019)

Teacher: There seems to be something wrong with the calculation of the fifth group? 40 ÷ (1 + 50%) = 40 ÷ 1.5 =? Will the result still be bigger than 40?

Student 1: I know, he is using multiplication when he found division did not work.

Teacher: Is that okay? What should I do?

Student 1: You can change the question to "There are 40 boxes of bananas in the supermarket, and the boxes of apples are 50% more than that of bananas. How many boxes are the apples?" The arithmetic expression is 40 × (1 + 50%) = 40 × 1.5 = 60 (boxes).

Student 2: It can also be "The supermarket has 40 boxes of bananas, which is 50% of the boxes of apples. How many boxes are there of apples?" The arithmetic expression is 40 ÷ 50% = 80 (boxes).

Teacher: We have sorted out these categories together just now. After the problem is posed, we can try to calculate it. If the numbers we use are found to be wrong, we can adjust it in time to avoid similar problems.

(Dialogue between teacher and student, 2019)

4.4. Guidelines for teachers to organize the class to modify the problems

For some problems that are hard to modify and not related to the teaching goal, teachers can choose not to modify these problems. Class time is an issue which is often reported by many teachers, so time should be spent on the problems that are connected to the teaching goal.

5. Practice: selecting the posed problems

Every class has its own teaching goals and time limits, and students' posed problems vary. Because of these variations and constraints, most of the selected problems should align the teaching goals. As discussed previously, in Zhang and Cai's (2021) analysis of 22 teaching cases, the teachers skipped the problems which were irrelevant to the teaching goal and mainly focused on moderately challenging problems in class. In Example 3, the topic how to write an arithmetic expression on division with remainder was taught. Several of the problems students posed are shown here. Considering the teaching goal was writing the arithmetic expression, the teacher moved the fourth and fifth questions which were more about calculation questions to the next class, and focused on the first three questions for the present class. This example shows the teachers select the problems considering the teaching goals and lesson arrangements.

Example 3:

Problem posing situation and prompt:

Select from these three shapes(△, □, ⬠). Use different numbers of sticks to represent them, and write arithmetic to represent the results of your drawing. Then pose at least two mathematics problems.

Students posed problems:

1. How to write the arithmetic using 8 small sticks to get the shape of a triangle?

2. How to write the arithmetic using 10 small sticks to get the shape of a triangle?

3. How to write the arithmetic using 11 small sticks to get the shape of a triangle?

4. How to work out the result of this arithmetic?

5. Why are there some extra sticks as a remainder?

(Problems posed by students, 2019)

5.1. Guidelines for selecting the posed problems

a) It is possible that students will pose high-quality, challenging problems that do not match the teaching goal for the specific lesson. Teachers may choose to skip these questions. When teachers choose to skip these questions, it is still important to let students know that these questions are not being ignored and will be discussed at a different time. In this way, students will not feel discouraged.

b) From observing problem-posing lessons, we have observed that students quickly lose interest in the activity if the selected posed problems are too easy for the whole class. Therefore, it is important to select problems that need some work to reach the final answer and spend less time on the easy problems.

c) It is sometimes the case that students' posed problems will not help the class reach the teaching goal. In this situation, the teacher might add some problems posed by themselves which they think will be important to help students stretch. In this way, the students' own posed problems will still be used, but the teaching goal can also be achieved.

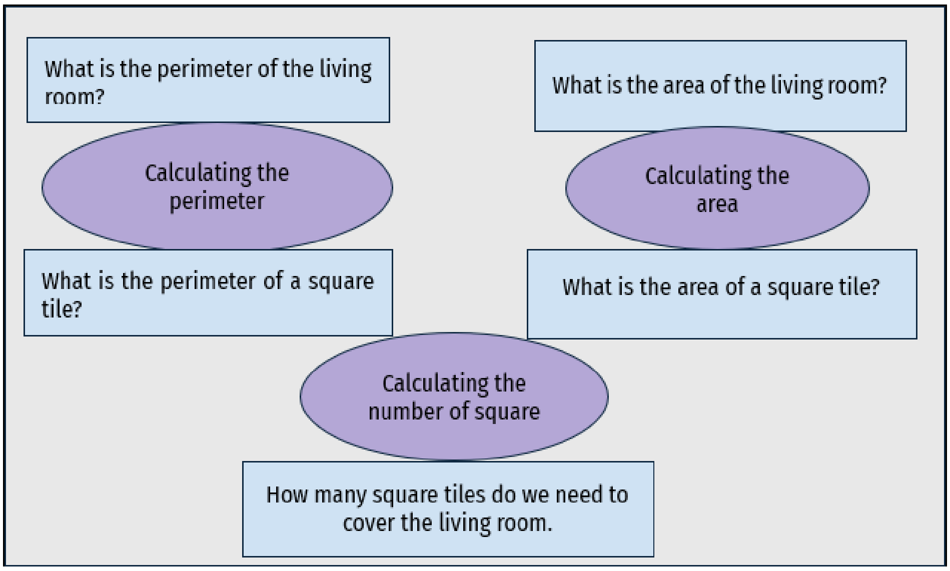
6. Practice: sequencing the posed problems

Sequencing is always an important practice in mathematics teaching. However, sequencing problems has rarely been discussed in prior problem-posing research. Stein *et al.* (2008) described practices for orchestrating productive mathematical discussion, emphasizing that teachers can maximize their chances of reaching the mathematical goal for the discussion by sequencing students' responses. Similarly, by sequencing students' posed problems, teachers will be more likely to achieve the mathematical goal of the problem-posing activity. In problem-posing activities, sequencing means that the teacher and students sequence the problems using some standards. Here are some frequently used standards for sequencing. For example, teachers might present the easiest problems to the students first and gradually increase the difficulty level to ensure that students construct knowledge and gain confidence from one problem to the next. Teachers and students might also sequence the problems by their level of interest, and they are more engaged in solving the problems they have more interest in. Teachers and students might sequence problems by order of dependency (e.g., Problem A's solution depends on Problem B's solution, and Problem B's solution depends on Problem C's solution).

In Example 4, the posed problems were analyzed and then classified into three groups by the level of interest and order of dependency. One group of problems was about calculating perimeter, one group was about calculating area, and another group was about calculating the number of square tiles (Figure 6). Then, students sequenced the problems under guidance from the teachers. They found that the problem about tiles was most interesting, and it relied on the answers for the area of a square tile and that of the living room. By sequencing the problems, they gained a better logic for solving the problems.

Example 4: Square Tiles

Figure 6: Square Tiles



Source: Researchers' archive (2019)

Teacher: Which problem do you think is the most interesting? Why do you think so?

Student 1: I think the problem about how many square tiles do we need to cover the living room is the most interesting one. The reason is that to solve this problem, we first need to calculate the area of the living room, and that of a square tile.

Student 2: The question about how many square tiles should be used also includes calculating the area of the living room, and that of a square tile.

(Dialogue between teacher and students, 2019)

6.1. Guidelines for sequencing the posed problems

a) When there is not enough time, teachers can directly sequence the problems and present them to students instead of waiting for students to sequence them.

7. Practice: solving the posed problems

When problem posing functions as a teaching method in the classroom, problem solving should be a necessary step. Singer and colleagues (2011) suggested that students can pose good problems, however their understanding of the problems and concepts can be superficial and limited. This indicates that problem solving might help with assessing students' understanding of the posed problems as well as their ability to solve the problems. Moreover, as studies (e.g., Bonotto; Dal Santo, 2014) suggested, problem solving after problem posing can help students better understand the initial problem situation, improve the quality of their posed problems, and increase their chances of posing further problems. Therefore, problem solving is considered as a necessary instructional practice to show students' understanding of the knowledge, and to what extent the students have solved the problem can be an indicator for assessing how much students have got command of the knowledge.

In a classroom setting, the problem solving can be performed in different group sizes, just as with problem posing. Students can work individually, in small groups or with a whole class. For example, Suarsana *et al.* 's (2019) study asked students to work in groups to pose problems and remain in their groups to solve other students' problems. Walkington's (2017) study asked students to either solve their own posed problems individually or work on problems they exchanged with other students. Zhang and Cai's (2021) study suggested that the whole class can work on the same problems together.

We continue Example 4 below, showing students thinking independently while solving posed problems in a whole-class environment. In this example, the teacher asked individual students to show their ideas and further asked if other students had different ideas. Students 1 and 2 presented different solutions to the whole class. Then, Student 3 summarized these two students' ideas. The teacher then asked students to think about why division was used for solving the problem, which the teacher thought to be important for students to notice. The individual work gave students the chance to think independently, and the group work gave them the chance to learn from others' strengths and build on other students' ideas.

Example 4 Continuation: Solution

Teacher: Can you solve the most interesting problem? Please try to do it.

Student 1: First, I got the area of the living room is 6 x 3 = 18 m², then I got the area of each square tile is 3 x 3 = 9 dm². Next, I convert 18 m² to 1800 dm². Last I divided 1800 by 9 equals 200 tiles. That is to say, there is a need for 200 tiles to cover the living room.

Teacher: Does everyone agree with his idea of solving this problem? Who has different ideas?

Student 2: I first converted 3 meters into 30 decimeters, and 6 meters into 60 decimeters, and then calculated the area of the living room. The following steps are the same.

Student 3: I got you. One idea of solving this problem is getting the area first and then converting the unit, and the other is converting the unit first and then finding the area.

> *Teacher: What you said is really good! Please think about this question: After we found the area of the living room and that of the square tiles, why did we use division to find the number of square tiles?*
>
> *Student 4: The question to find out how many square tiles are needed is to find out how many nines are included in 1800. The meaning of division is to find how many of one number are included in another number. Therefore, we used division to do the calculation.*
>
> *Student 5: I think the area of the living room, which is 1800 square decimeters, is the total number; the area of the square tile, which is 9 square decimeters, is the number of each share. Finding the number of square tiles is to find the number of shares. Using the total ÷ the number of each share = the number of shares. Therefore, we use division to find the number of tiles.*
>
> *Teacher: That is to say, the number of shares here is the number of tiles, the total is the total area of the living room, and the number of each share is the area of each tile. Therefore, the number of tiles = the total area ÷ the small area.*
>
> *(Dialogue between teacher and students, 2019)*

7.1. Guidelines for sequencing the problems

Teachers have worked hard in the prior steps and sub-steps (i.e., present the problem posing situation and the prompts; facilitate students to pose problems; analyze, select, and sequence the problems). We noticed that in some classes, in this last step, the teachers showed the problems in the textbook directly to students. However, we consider it is important to solve the problems from the selected and sequenced ones, which indicates that students are working on their own problems. Also, after the prior steps, the posed problems also align with the teaching goals.

8. Conclusion

The instructional model we have described in this chapter provides teachers with a practical and effective approach to initiating problem-posing activities in their classroom. This model can be useful for teachers at all levels, from K-12 teachers to teacher educators who are looking to incorporate problem posing in their instruction. This instructional model can provide teachers with detailed steps to design problem-posing tasks or modify existing tasks to make them more problem-posing focused as well as provide teachers with specific practices to help them better handle students' posed problems. By implementing this teaching model, teachers can not only address the time constraints they may face but also achieve their teaching goals more effectively. Additionally, it can provide pedagogical advantages to teachers, thereby building teachers' confidence and promoting their efficacy in teaching through problem posing. Furthermore, when teachers are more capable of dealing with students' posed problems, they can create a more positive and supportive learning environment, which can also promote students' learning.

Challenges remain in implementing this instructional model. As noted for each practice. To implement problem posing into a typical classroom where time is always said to be a constraint. Teachers are expected to have a strong understanding of their students' prior knowledge and experience as well as a deep knowledge of the content they are teaching, so that they can quickly analyze the posed problems during Step 3 and determine how to organize subsequent activities. It is our hope that teachers can learn more about teaching through problem posing by utilizing this model in their problem-posing lessons. We need more teaching cases to illustrate and validate this model, thereby improving the model by adding more specific practices and possible steps.

Acknowledgements

Preparation of this manuscript was supported by a grant from the USA National Science Foundation (DRL- 2101552). Any opinions expressed herein are those of the authors and do not necessarily represent the views of funding agency.

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Editorial History
Received on 18/06/2024.
Accepted on 30/07/2024.
Published on 13/08/2024.

How to cite – ABNT
MA, Yue; CAI, Jinfa. Teaching Mathematics through Problem Posing: four practices for handling students' posed problems. **REVEMOP**, Ouro Petro/MG, Brasil, v. 6, e2024009, 2024. https://doi.org/10.33532/revemop.e2024009

How to cite – APA
Ma, Y., & Cai, J. (2024). Teaching Mathematics through Problem Posing: four practices for handling students' posed problems. *REVEMOP*, *6*, e2024009. https://doi.org/10.33532/revemop.e2024009