Existence of Solutions For a Class of Quasilinear Elliptic Equation in $\mathbb{R}^N$ With Vanishing Potentials

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We establish the existence of positive solution for the following class of quasilinear elliptic problem

$$(P) \qquad \begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = f(u) \text{ in } \mathbb{R}^N, \\ u > 0, \text{ in } \mathbb{R}^N; \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases}$$

where Δ_p is the p-Laplacian, V is a bounded non negative vanishing potential and f has a subcritical growth at infinity. The technique used here is a truncation argument together with the variational approach. We impose the following hypothesis on our functions V and f:

$V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function verifying

$(V_1) : V(x) \geq 0, \forall\, x \in \mathbb{R}^N$

$(V_2) : V(x) \leq V_\infty, \forall\, x \in B_1(0)$ and for some $V_\infty > 0$.

$(V_3) : \exists\ \Lambda > 0$ and $\exists\ R > 1$ such that $\displaystyle\inf_{|x|>R} \left(\frac{|x|}{R}\right)^{\frac{p^2}{p-1}} V(x) \geq \Lambda.$

$f : \mathbb{R} \to \mathbb{R}$ is continuous function verifying

$(f_1) : \displaystyle\limsup_{s\to 0^+} \frac{sf(s)}{s^{p^*}} = 0$, where $p^* = \frac{pN}{N-p}$, $N > p > 1$.

$(f_2) : \exists\ \alpha \in (p, p^*)$ such that $\displaystyle\lim_{s\to\infty} \frac{sf(s)}{s^\alpha} = 0.$

$(f_3) : \exists\ \theta > p$ such that $\theta F(s) \leq sf(s), \forall s > 0.$

Equations involving the p-Laplacian operator appear in many problems of nonlinear diffusion. Just to mention, in nonlinear optics, plasma physics, condensed matter physics and in modeling problems in non Newtonian fluids. For more information on the physical background we refer to [15]. For the case when $p = 2$ and the potential is bounded from below by a positive constant $V_0 > 0$, we cite [3, 4, 5, 8, 9, 11, 13, 20, 24, 26, 27, 28, 29], and references therein . In [18], in addition to the above assumptions, the authors consider a local condition, namely, $\min_{x\in\overline{\Omega}} V < \min_{x\in\partial\Omega} V$, where $\Omega \subset \mathbb{R}^N$ is a open bounded set, instead of the global condition imposed by Rabinowitz in [20]. When $p \neq 2$, see [6, 10, 12]. When $p = 2$ and V is the zero mass case, that is $\lim_{|x|\to\infty} V(x) = 0$, we cite [1, 2, 25] and the recent paper [7] by Alves and Souto. The result presented here for $1 < p < N$ extends that one in [7] for $p = 2$. In [7] the presence of Hilbertian structure and some compact embeddings provide the convergence of the gradient. In the case studied here, we loose this structure and we do not obtain the convergence so directly. To overcome this problem we adapt a result of [22, proposition 1.5, page 22], whose ideas come from [17, 19]. Together with this difficulty there are others. For instance, in the present situation our space is no longer Hilbert which forces us to obtain new estimates. Now we state the main result of this work.

Teorema : Suppose that V and f satisfy, respectively, $(V_1)-(V_3)$ and $(f_1)-(f_3)$. Then there is a constant $\Lambda^* = \Lambda^*(V_\infty, \theta, p, c_0) > 0$ such that problem (P) has a positive solution, for all $\Lambda \geq \Lambda^*$.

In order to achieve this, we first build an auxiliary problem (AP). Then we solve the problem (AP) using variational methods and to finish we show that the solution of (AP) is also a solution of (P).

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