Hipercontractivity and Hamilton-Jacobi equations

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Introduction

Let be the following Hamilton-Jacobi equation

$$\begin{aligned} u_t + \frac{1}{2}||\nabla u||^2 = 0 \qquad \text{in } (0,\infty)\times R^d, \\ u(0,x) = g(x) \quad \text{on } R^d. \end{aligned} \tag{1}$$

where g is Lipschitz on R^d. It is known that the Hopf-Lax formula, defined as

$$u(t,x) = \min_{y\in R^d}\left\{\frac{||x-y||^2}{2t} + g(y)\right\} \tag{2}$$

solve this problem *almost ever* in $(0,\infty)\times R^d$, and the solution is a Lipschitz continuous function. Moreover, this formula defines a semigroup $(H_t g)_{t\geq 0}$, and this semigroup has the following property: *Let μ be a probability measure, absolutely continuous with respect to Lebesgue measure and that for some $\rho > 0$ and all smooth enough functions f on R^d,*

$$Ent_\mu(f^2) \leq 2||\nabla f||_2^2. \tag{3}$$

Then, for every bounded measurable function f on R^d, every $t \geq 0$, and every $a \in R$

$$||e^{Q_t f}||_{a+\rho t} \leq ||e^f||_a. \tag{4}$$

Conversely, if (4) *holds for all $t \geq 0$ and some $a \neq 0$ then,* (3) *holds.* The functional $Ent_\mu(f)$ is defined as

$$Ent_\mu(f) := \int_{R^d} f log(f) d\mu - (\int_{R^d} f d\mu) log(\int_{R^d} f d\mu).$$

This functional is known as the *Entropy* of the function f with respect to the measure μ, and the inequality (3) is known as the Log- Sobolev inequality. This equivalence was proved in [1], following the ideas of [4], where was proved a equivalence between the Log-Sobolev inequality and the heat equation (for a probability measure μ). When a semigroup satisfies the inequality (4), we say that this semigroup is *hipercontractive.* This idea appeared first in [5], and since then, this idea had been used in differrent fields, as in quantum mechanics, or to obtain more regularity in the case of the solution of Hamilton-Jacobi equation.

Main result

In this work, we study the following *weighted* Hamilton-Jacobi equation

$$u_t(t,x) + \frac{1}{2}w(t,x)^2||\nabla u||^2 = 0; \quad (0,\infty)\times R^d;$$
$$u(0,x) = g(x); \quad R^d,$$

We give some conditions on g and give a Hopf-Lax formula for this problem. As an important example, we obtain a solution for the following Cauchy problem

$$u_t(t,x) + \frac{1}{2}w(t,x)^2||\nabla u||^2 = 0; \quad (0,\infty)\times R^d;$$
$$u(0,x) = ||x||; \quad R^d,$$

with

$$w(t,x) = \begin{cases} At^{\frac{1-\alpha}{2\alpha-1}} & ; \quad ||x|| > \left(\frac{t}{\alpha}\right)^{\frac{1}{2\alpha-1}}, \\ \frac{\sqrt{2}}{\alpha||x||^{\alpha-1}} & ; \quad ||x|| \leq \left(\frac{t}{\alpha}\right)^{\frac{1}{2\alpha-1}} \end{cases}$$

where A is a real constant.

Referências

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