p-Laplacian with fast growing gradient

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We consider the existence of positive solutions for the Dirichlet problem in two positive parameters λ and β

$$\begin{cases} -\Delta_p u = \lambda h(x,u) + \beta f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where h and f are continuous nonlinearities satisfying

(H1) $0 \le \omega_1(x)u^{q-1} \le h(x,u) \le \omega_2(x)u^{q-1}$ $(1 < q < p)$;

(H2) $0 \le f(x,u,v) \le \omega_3(x)u^a|v|^b$,

where ω_i, $1 \le i \le 3$, are nonnegative weights in the smooth and bounded domain $\Omega \subset \mathbb{R}^N$ and $a, b > 0$. We prove the existence of a region $\mathcal{D}$ in the $\lambda\beta$-plane where the Dirichlet problem has at least one positive solution.
Our proof rests on a version of the classical result of Tolksdorf and Lieberman, which provides *a priori* bounds for the gradient.
Theorem [Tolksdorf-Lieberman] *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain and that $g \in L^\infty(\Omega)$. Then there exists a positive constant $\mathcal{K}$, depending only on p and Ω, such that*

$$\|\nabla u\|_\infty \le \mathcal{K}\, \|g\|_\infty^{\frac{1}{p-1}}$$

where $u \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ is the only weak solution of

$$\begin{cases} -\Delta_p u = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let u_1 be the first positive eigenfunction of the p-Laplacian with weight ω_1, that is,

$$\begin{cases} -\Delta_p u_1 = \lambda_1 \omega_1 u_1^{p-1} & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\|u_1\|_\infty = 1$ and let $\phi \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ be the solution of the problem

$$\begin{cases} -\Delta_p \phi = \omega & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\omega(x) := \max_{i\in\{1,2,3\}} w_i(x)$.
We denote

$$\gamma := \frac{\mathcal{K}\, \|\omega\|_\infty^{\frac{1}{p-1}}}{\|\phi\|_\infty},$$

where $\mathcal{K}$ stands for the constant of the result of Tolksdorf and Lieberman.

Our main result is
Theorem: *Assume that h and f are continuous and satisfy* (H1) *and* (H2). *There exists a region $\mathcal{D}$ in the $\lambda\beta$-plane such that if $(\lambda, \beta) \in \mathcal{D}$ the Dirichlet problem (P) has at least one positive solution u satisfying, for some positive constants ϵ and M:*

$$\epsilon u_1 \leq u \leq \frac{M\phi}{\|\phi\|_\infty} \qquad \text{and} \qquad \|\nabla u\|_\infty \leq \frac{\mathcal{K}\, \|\omega\|_\infty^{\frac{1}{p-1}}\, M}{\|\phi\|_\infty}.$$

To prove our result we define for each $u \in C^1$,

$$F^u(x,\xi) := \lambda\omega_1(x)\xi^{q-1} + \lambda\left(h(x,u(x)) - \omega_1(x)u(x)^{q-1}\right) + \beta f(x,u(x),\nabla u(x)) \in C(\overline{\Omega}).$$

We observe that $F^u(x,u) = \lambda h(x,u) + \beta f(x,u,\nabla u)$.
Lemma: *There exist positive constants ϵ and M, and a region $\mathcal{D}$ in the $\lambda\beta$-plane such that, if $(\lambda,\beta) \in \mathcal{D}$ and*

$$u \in \mathcal{F} := \left\{ u \in C^1(\overline{\Omega}) : 0 \leq u \leq \frac{M\phi}{\|\phi\|_\infty} \quad \textit{and} \quad \|\nabla u\|_\infty \leq \gamma M \right\},$$

then the Dirichlet problem

$$\begin{cases} -\Delta_p U &= F^u(x,U) \quad \textit{in } \Omega \\ \quad U &= 0 \qquad\qquad \textit{on } \partial\Omega. \end{cases} \tag{P$_a$}$$

has $\underline{u} := \epsilon u_1$ and $\overline{u} = \frac{M\phi}{\|\phi\|_\infty}$ as an ordered pair of sub- and super-solution and a unique positive solution U.
The lemma is proved by applying the sub- and super-solution method, inspired by the seminal paper of Ambrosetti, Brezis and Cerami [1]. The uniqueness of U follows from well-known results proved in [3].
We define the operator

$$\begin{array}{rcl} T\colon \mathcal{F} \subset C^1(\overline{\Omega}) & \longrightarrow & C^{1,\alpha}(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \subset C^1(\overline{\Omega}) \\ u & \longrightarrow & U, \end{array}.$$

The existence of a fixed point of T follows from Schauder's fixed point theorem.

Referências

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