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On discontinuous differential equations

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Abstract

The paper studies and compare generalized solutions to discontinuous differential equations. This way, are studied and compared the Euler, Filippov, Hermes, Krasovskii, and Sentis generalized solutions to discontinuous differential equations. In particular, are studied relations between Euler and Hermes solutions. In addition, it is proved that Hermes solutions satisfy some properties that are analogous to the properties satisfied by Euler solutions.

Keywords

Differential inclusions, Discontinuous differential equations, Generalized solutions.

1 Introduction

In this paper are studied ordinary differential equation with a discontinuous right hand side. Such that differential equations are named discontinuous differential equations. Here both the autonomous ordinary differential equations as well as nonautonomous ordinary differential equations will be studied. This way, are studied generalized solutions to discontinuous differential equations.

The study of discontinuous differential equations can be found, for example, in [3, 4, 5, 6, 7, 8, 10, 11, 13, 14]. In particular, Euler solutions were considered in [3, 4, 6, 7, 8, 14] while Hermes solutions were treated in [3, 8, 10, 14]. On the other hand, Filippov solutions were studied by [3, 5, 6, 8, 10, 13, 14] while Krasovskii solutions were studied by [3, 6, 8, 10, 14]. Finally, Sentis solutions were named by [3] and introduced by [13]. Sentis solutions were also considered by [4, 5, 11].

The present paper studies generalized solutions for the initial value problems

$$\dot{x}(t) = f(t, x(t)), \quad x(a) = x_0 \tag{1}$$

and

$$\dot{x}(t) = g(x(t)), \quad x(a) = x_0 \tag{2}$$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$. Thus, are studied properties of the Euler and Hermes solutions to Eq. (1). On the other hand, the study of Filippov, Krasovskii and Sentis solutions are related to Eq. (2). If f is a continuous function, then in [7] it is proved that any Euler solution x to Eq. (1) is continuously differentiable on (a, b) and satisfies $\dot{x}(t) = f(t, x(t))\ \forall t \in (a, b)$. This property for Euler solutions is stated in part (c) of the theorem [[7],1.7. Theorem]. An analogous result is proved here for Hermes solutions. The theorem [[7],1.7. Theorem] deals with properties for Euler solutions. Analogues to parts (a) and (b) of this theorem for Hermes solutions to Eq. (1) are also obtained here.

Motivated by [4] by the study on relationships between generalized solutions to Eq. (2), the present work also studies relationships between the Euler, Filippov, Hermes, Krasovskii, and Sentis generalized solutions to Eq. (2). It is convenient to highlight the novelty of study in present work involving relationships between the Euler, Hermes, Krasovskii, and Sentis solutions to Eq. (2). In particular, it is proved that a Sentis solution is a Hermes solution.

2 Preliminaries

In this section are considered concepts and results that will be used in the development of the present work.

2.1 Lebesgue measure and integral

Here are reminded basic concepts of measure theory. A more complete approach to Lebesgue measure and integral, can be found in [12].

Definition 2.1. *A function $f : [a, b] \to \mathbb{R}^n$ is said to be Lebesgue measurable if for all open set $V \subset \mathbb{R}^n$, the set*

$$f^{-1}(V) = \{t \in [a, b] : f(t) \in V\}$$

is Lebesgue measurable.

Let $I \subset \mathbb{R}$ be an interval. It is said that a statement P holds almost everywhere (a.e.) on I, if the set N given by

$$N = \{t \in I : P \text{ does not hold at } t\}$$

has Lebesgue measure zero.

The Banach space of Lebesgue integrable functions $x : [a, b] \to \mathbb{R}^n$ with the norm

$$\|x(\cdot)\|_{L_1} = \int_a^b |x(t)|dt$$

will be denoted by $L_1([a, b])$.

2.2 Absolutely continuous functions

The definition of absolutely continuous function can be found in [2, 15].

Definition 2.2. *A function $x : [a, b] \to \mathbb{R}^n$ is called absolutely continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any countable collection of disjoint subintervals $[a_k, b_k]$ of $[a, b]$ such that*

$$\sum (b_k - a_k) < \delta,$$

we have

$$\sum |x(b_k) - x(a_k)| < \varepsilon.$$

Below is a result considered in [2] for absolutely continuous functions.

Theorem 2.1. *A continuous function is the integral of its derivative if and only if it is an absolutely continuous function.*

As discussed in [15], an absolutely continuous function $x : [a, b] \to \mathbb{R}^n$ is differentiable almost everywhere, and its derivative $\dot{x}(\cdot)$ is a Lebesgue integrable function. Moreover, the Newton-Leibniz formula is true; that is,

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} \dot{x}(t)dt$$

for all $t_1, t_2 \in [a, b]$, $t_1 < t_2$. Hence any absolutely continuous function $x : [a, b] \to \mathbb{R}^n$ can be represented in the form

$$x(t) = x(a) + \int_a^t \dot{x}(s)ds.$$

We will refer to any absolutely continuous function $x : [a, b] \to \mathbb{R}^n$ as an arc on $[a, b]$.

We will use the following result, known as Gronwall's Lemma and discussed in

[7].

Proposition 2.1. *Let x be an arc on $[a, b]$ which satisfies*

$$\|\dot{x}(t)\| \leq \gamma \|x(t)\| + c(t) \;\; a.e., \quad t \in [a, b],$$

where γ is a nonnegative constant and where $c(\cdot) \in L_1([a, b])$ is a nonnegative function. Then, for all $t \in [a, b]$, we have

$$\|x(t) - x(a)\| \leq (e^{\gamma(t-a)} - 1)\|x(a)\| + \int_a^t e^{\gamma(t-s)} c(s) ds.$$

If the function c is constant and $\gamma > 0$, this becomes

$$\|x(t) - x(a)\| \leq (e^{\gamma(t-a)} - 1)(\|x(a)\| + c/\gamma).$$

2.3 First fundamental theorem of calculus

We will use the first fundamental theorem of calculus, as stated below.

Theorem 2.2 ([1])**.** *Let f be a function that is integrable on $[a, x]$ for each x in $[a, b]$. Let c be such that $a \leq c \leq b$ and define a new function A as follows:*

$$A(x) = \int_c^x f(t) dt \quad if \quad a \leq x \leq b.$$

Then the derivative $A'(x)$ exists at each point x in the open interval (a, b) where f is continuous, and for such x we have

$$A'(x) = f(x).$$

3 Generalized solutions to Eqs. (1) and (2)

In this section, we find the main results of the present study. The contributions of the paper to the theory of differential equations with discontinuous right-hand sides are stated in Theorem 3.2, Proposition 3.2 and Lemma 3.1.

3.1 Euler and Hermes solutions to Eq. (1)

The following are the Euler and Hermes solutions to (1). Initially the Euler solutions are defined as in [7]. Thus, let

$$\pi = \{t_0, t_1, ..., t_{N-1}, t_N\}$$

be a partition of $[a, b]$, where $t_0 = a$ and $t_N = b$. The diameter of the partition π is given by

$$\mu_\pi := \max\{t_i - t_{i-1} : 1 \leq i \leq N\}.$$

Definition 3.1. *We define the Euler polygonal arc for Eq. (1) and corresponding to the partition π, by the arc $x_\pi : [a, b] \to \mathbb{R}^n$ given by:*

$$x_\pi(t_0) = x_0, \;\; x_\pi(t) = x_0 + (t - t_0)f(t_0, x_0), \;\; t \in [t_0, t_1]$$
$$x_\pi(t_1) = x_1, \;\; x_\pi(t) = x_1 + (t - t_1)f(t_1, x_1), \;\; t \in [t_1, t_2]$$

and by induction

$$x_\pi(t_i) = x_i, \;\; x_\pi(t) = x_i + (t - t_i)f(t_i, x_i), \;\; t \in [t_i, t_{i+1}]$$

when $i \in \{0, 1, \ldots, N - 1\}$.

Definition 3.2. *We say that the arc $x : [a, b] \to \mathbb{R}^n$ is an Euler solution for (1), if x is the uniform limit of Euler polygonal arcs x_{π_j}, corresponding to some sequence π_j such that $\mu_{\pi_j} \to 0$.*

Below we state the theorem [[7],1.7. Theorem].

Theorem 3.1 ([7])**.** *Suppose that for positive constants γ and c and for all $(t, x) \in [a, b] \times \mathbb{R}^n$, we have the linear growth condition*

$$\|f(t, x)\| \leq \gamma\|x\| + c,$$

where f is otherwise arbitrary. Then:

(*a*) *At least one Euler solution x to the initial-value problem (1) exists on $[a, b]$, and any Euler solution is Lipschitz.*

(*b*) *Any Euler arc x for f on $[a, b]$ satisfies*

$$\|x(t) - x(a)\| \leq (t - a)e^{\gamma(t-a)}(c + \gamma\|x(a)\|), \quad a \leq t \leq b.$$

(c) *If f is continuous, then any Euler arc x for f on $[a,b]$ is continuously differentiable on (a,b) and satisfies $\dot{x}(t) = f(t, x(t))\ \forall t \in (a,b)$.*

Examples of Euler solutions are given below. Thus, let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(t,x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

and let $x_0 = 0$. Given a partition $\pi_j = \{t_0^{(j)}, t_1^{(j)}, ..., t_{N_j}^{(j)}\}$ of $[0,1]$, the Euler polygonal arc x_{π_j} is given by

$$x_{\pi_j}(t) = t, \quad t \in [t_0^{(j)}, t_1^{(j)}]$$
$$x_{\pi_j}(t) = t_1^{(j)}, \quad t \in [t_i^{(j)}, t_{i+1}^{(j)}]$$

when $i \in \{1, ..., N_j - 1\}$. Since $|x_{\pi_j}(t)| \leq |t_1^{(j)}|$ for each $t \in [0,1]$, and since $t_1^{(j)} \to 0$ whenever $j \to \infty$, then $x(t) = 0$ is the only Euler solution for (1). Now consider the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ given by

$$f(t,x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$$

and let $x_0 = 0$. For every partition π of $[0,1]$ we have $x_\pi = 0$. Therefore, $x(t) = 0$ is the only Euler solution for (1). Suppose now that $x_0 = 1$. In this case, for every partition π of $[0,1]$ it follows that $x_\pi = 1 + t$. Thus, $x(t) = 1 + t$ is the only Euler solution for (1).

To define Hermes solutions we must first define Carathéodory solutions.

Definition 3.3. *An arc $x : [a,b] \to \mathbb{R}^n$ is a Carathéodory solution to (1) if, and only if, x satisfies the differential equation given in Eq. (1) for a.e. $t \in [a,b]$ and $x(a) = x_0$.*

Below we define Hermes solutions as [8].

Definition 3.4. *Let $x : [a,b] \to \mathbb{R}^n$ be an arc. We say that x is a Hermes solution to Eq. (1) if, and only if, there exist functions Lebesgue measurable $p_j : [a,b] \to \mathbb{R}^n$ and Carathéodory solutions x_j to the initial value problem*

$$\dot{y}(t) = f(t, y(t) + p_j(t)), \quad y(a) = x_0 \tag{3}$$

such that $p_j \rightrightarrows 0$ and $x_j \rightrightarrows x$.

The notations $p_j \rightrightarrows 0$ and $x_j \rightrightarrows x$ means uniform convergence. We also note that an Euler solution is a Hermes solution, as stated below.

Proposition 3.1. *Suppose that there are positive constants γ and c such that*

$$\|f(t,x)\| \leq \gamma\|x\| + c \tag{4}$$

for every $(t,x) \in [a,b] \times \mathbb{R}^n$. Then every Euler solution of Eq. (1) is a Hermes solution.

The proof of the above result can be found in [14]. An example of a Hermes solution is considered below. For this, consider again the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ given by

$$f(t,x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$$

and let $x_0 = 0$. If $p_j(t) = 1/j$ and $x_j(t) = t$, then x_j is a Carathéodory solution to Eq. (3) for every $j \in \mathbb{N} \setminus \{0\}$. Therefore $x(t) = t$ is a Hermes solution to Eq. (1). Since in this case $x(t) = 0$ is the only Euler solution to Eq. (1), we conclude that in general a Hermes solution is not an Euler solution.

In the theorem below we get properties for Hermes solutions that make an analogy with the theorem [[7],1.7. Theorem].

Theorem 3.2. *Suppose f is Lebesgue measurable in t for each x fixed. Assume also that f satisfies the linear growth condition given in Eq. (4). Then:*

(*a*) *There exists at least one Hermes solution y to Eq. (1). In addition, any Hermes solution is Lipschitz continuous.*

(*b*) *There exists a positive constant K such that every Hermes solution satisfies*

$$\|y(t) - y(a)\| \leq (e^{\gamma(t-a)} - 1)(\|y(a)\| + K/\gamma)$$

for all $t \in [a,b]$.

(*c*) *If f is a continuous function, then any Hermes solution y to Eq. (1) is continuously differentiable on (a,b) and satisfies $\dot{y}(t) = f(t,y(t))\ \forall t \in (a,b)$.*

Proof. The existence of at least one Hermes solution y is established in the theorem [[8], Theorem 6]. If y is a Hermes solution to Eq. (1), then there exist Lebesgue measurable functions $p_j : [a,b] \to \mathbb{R}^n$ and Carathéodory solutions y_j of Eq. (3) such that $p_j \rightrightarrows 0$

and $y_j \rightrightarrows y$. Then there exist $N \in \mathbb{N}$ and $c_1 > 0$ satisfying $\|p_j(t)\| \leq 1$ and $\|y_j(t)\| \leq c_1$ for all $t \in [a,b]$, whenever $j \geq N$. Thus, if $L = \gamma c_1 + \gamma + c$, then

$$\begin{aligned} \|\dot{y_j}(t)\| &= \|f(t, y_j(t) + p_j(t))\| \leq \gamma \|y_j(t) + p_j(t)\| + c \\ &\leq \gamma \|y_j(t)\| + \gamma \|p_j(t)\| + c \leq \gamma c_1 + \gamma + c \\ &= L \;\; a.e., \quad t \in [a,b] \end{aligned}$$

hence y_j is Lipschitz continuous with Lipschitz constant L. Thus, y is Lipschitz continuous with Lipschitz constant L. This proves item (a).

Let y and y_j be as in the proof of item (a). We saw that

$$\|\dot{y_j}(t)\| \leq \gamma \|y_j(t)\| + \gamma + c \;\; a.e., \quad t \in [a,b]$$

for every $j \geq N$. If $K = \gamma + c$, we have

$$\|\dot{y_j}(t)\| \leq \gamma \|y_j(t)\| + K \;\; a.e., \quad t \in [a,b]$$

and from Proposition 2.1 it follows that

$$\|y_j(t) - y_j(a)\| \leq (e^{\gamma(t-a)} - 1)(\|y(a)\| + K/\gamma)$$

for all $t \in [a,b]$. Therefore

$$\|y(t) - y(a)\| \leq (e^{\gamma(t-a)} - 1)(\|y(a)\| + K/\gamma)$$

for all $t \in [a,b]$, proving item (b).

Again, let y and y_j be as in the proof of item (a). Since a continuous function in $\mathbb{R}^{n+1}$ is uniformly continuous in compact sets, for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying

$$\|f(t,z) - f(t,w)\| < \varepsilon$$

whenever $\|z\| \leq c_1 + 1$, $\|w\| \leq c_1 + 1$ and $\|z - w\| < \delta$. Hence, there exists $J \in \mathbb{N}$ such that

$$\|f(t, y_j(t) + p_j(t)) - f(t, y_j(t))\| < \varepsilon$$

for all $j \geq J$. In this way, if $j \geq J$ we have

$$\begin{aligned}
&\|y_j(t) - y_j(a) - \int_a^t f(\tau, y_j(\tau))d\tau\| \\
&= \| \int_a^t \{\dot{y_j}(\tau) - f(\tau, y_j(\tau))\}d\tau\| \\
&= \| \int_a^t \{f(\tau, y_j(\tau) + p_j(\tau)) - f(\tau, y_j(\tau))\}d\tau\| \\
&\leq \varepsilon(t-a) \leq \varepsilon(b-a).
\end{aligned}$$

Taking $j \to \infty$ we may conclude that

$$\|y(t) - y(a) - \int_a^t f(\tau, y(\tau))d\tau\| \leq \varepsilon(b-a).$$

Since ε is arbitrary, we get

$$y(t) = y(a) + \int_a^t f(\tau, y(\tau))d\tau$$

and from Theorem 2.2 y is continuously differentiable on (a, b) and satisfies $\dot{y}(t) = f(t, y(t))$ for all $t \in (a, b)$. Thus, we conclude the proof of item (c). □

3.2 Filippov and Krasovskii solutions to Eq. (2)

In this work, it is assumed that the vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ is Lebesgue measurable and locally bounded to define the Filippov and Krasovskii solutions to Eq. (2).

The notions of Filippov and Krasovskii solutions to Eq. (2) consists of replacing the differential equation given in Eq. (2) by a suitable differential inclusion:

$$\dot{x}(t) \in G(x(t)). \tag{5}$$

By ordinary solution of Eq. (5) we mean any arc $x : [a, b] \to \mathbb{R}^n$ satisfiyng $\dot{x}(t) \in G(x(t))$ a.e. on the interval $[a, b]$. Filippov solutions to Eq. (2) are the ordinary solutions of Eq. (5) obeying $x(a) = x_0$, where

$$G(x) = G_F(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{co}\{g(\mathcal{B}(x, \delta) \setminus N)\}$$

while Krasovskii solutions of Eq. (2) are the ordinary solutions of Eq. (5) obeying $x(a) = x_0$, where

$$G(x) = G_K(x) = \bigcap_{\delta>0} \overline{co}\{g(\mathcal{B}(x,\delta))\}$$

where μ is the Lebesgue measure of $\mathbb{R}^n$, $\overline{co}$ denotes the closure of the convex hull, and $\mathcal{B}(x,r)$ is the open ball of radius r centered at x. Since $G_F(x) \subset G_K(x)$, every Filippov solution is a Krasovskii solution. The set of Filippov (respectively, Krasovskii) solutions of Eq. (2) will be denoted by $\mathcal{F}$ (respectively, $\mathcal{K}$). Furthermore, we denote the sets of Euler, Hermes, and Carathéodory solutions to Eq. (2) respectively by $\mathcal{E}$, $\mathcal{H}$, and $\mathcal{C}$.

Since g is measurable and locally bounded, then the set valued map $G_F(x)$ is upper semicontinuous, locally bounded, compact and convex valued. The same holds true for $G_K(x)$. It follows that the Eq. (2) has a Filippov solution (and hence a Krasowskii solution) on some interval $[a, c]$ ([9]).

It is well known that $G_K(x) = G_F(x) = g(x)$ whenever the function g is continuous at x. Then $\mathcal{F} = \mathcal{K} = \mathcal{C}$ when the function g is continuous.

As we can see in corollary [[10],Corollary 5.6.], $\mathcal{K} = \mathcal{H}$.

Now consider the differential equation

$$\dot{x}(t) = \frac{3}{2}x^{1/3}, x(0) = 0 \tag{6}$$

on the interval $[0, 1]$. The Eq. (6) has three distinct Carathéodory solutions: $x(t) = t^{3/2}$, $x(t) = -t^{3/2}$ and $x(t) = 0$. However the only Euler solution to Eq. (6) is $x(t) = 0$. Since $g(x) = \frac{3}{2}x^{1/3}$ is a continuous function, we may conclude that $\mathcal{F} = \mathcal{K} = \mathcal{H} = \mathcal{C}$. Therefore, even in the case where g is a continuous function, in general a Hermes solution is not an Euler solution. Thus, we get the following proposition.

Proposition 3.2. *Suppose that the function $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Then $\mathcal{E} \subset \mathcal{H} = \mathcal{K} = \mathcal{F} = \mathcal{C}$.*

3.3 Sentis solutions to Eq. (2)

Sentis solutions to Eq. (2) will be defined as g-solutions of Eq. (5). Thus, g-solutions are defined below. He we also assumed that the vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ is Lebesgue measurable and locally bounded.

According to [13] we replace the Eq. (2) by a differential inclusion given in Eq. (5)

where

$$G(x) = G_S(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\{g(\mathcal{B}(x,\delta) \setminus N)\}}. \tag{7}$$

The set valued map $G_S(x)$ is upper semicontinuous, locally bounded and compact (but in general not convex) valued.

Let x_0 be a fixed point of $\mathbb{R}^n$. Given $m \in \mathbb{N}$, consider a partition of $[a,b]$

$$a = t_{m,0} < t_{m,1} < \cdots < t_{m,k_m-1} < t_{m,k_m} = b$$

where k_m is some positive integer, and let

$$l_m = \max\{t_{m,i+1} - t_{m,i}, i = 0, \dots, k_m - 1\}.$$

Then, for each $i = 0, \dots, k_m - 1$ take $\varepsilon_{m,i} \in \mathbb{R}^n$ and construct a piecewise linear function $\psi_m(t)$ on the interval $[a,b]$ satisfying $\psi_m(t_{m,0}) = x_0$, and

$$\psi_m(t_{m,i+1}) = \psi_m(t_{m,i}) + v_{m,i}(t_{m,i+1} - t_{m,i}) + \varepsilon_{m,i}$$

where $v_{m,i}$ is an arbitrary element in $G(\psi_m(t_{m,i}))$. The function $\psi_m(t)$ will be called a polygonal approximation.

Definition 3.5. *A function $\varphi : [a,b] \to \mathbb{R}^n$ is a g-solution of Eq. (5) if for each $\sigma > 0$ there exists an integer m and a polygonal approximation obeying*

$$\|\varphi(t) - \psi_m(t)\| < \sigma \quad \forall t \in [a,b]$$

$0 < l_m < \sigma$ *and* $0 \leq \sum_{i=0}^{k_m-1} \|\varepsilon_{m,i}\| < \sigma.$

By construction, $\varphi(a) = x_0$. It is proved in [13] that if $G(x)$ is upper semicontinuous, compact valued and locally bounded, then for each x_0 there exist $b > a$ and a g-solution $\varphi : [a,b] \to \mathbb{R}^n$ with $\varphi(a) = x_0$. It is also proved in [13] that ordinary solutions of Eq. (5), when they exists, are g-solutions. Other properties concerning g-solutions can be founded in [3] and [13].

We say that a function $\varphi(t)$ is a Sentis solution of Eq. (2) if it is a g-solution of Eq. (5) with $G(x) = G_S(x)$. The set of Sentis solutions of Eq. (2) will be denoted by $\mathcal{S}$.

Proposition 3.3 ([13]). *Suppose that the function $g : \mathbb{R}^n \to \mathbb{R}^n$ is Lebesgue measurable and* *locally bounded. Then every Sentis solution of Eq.* (2) *is a Filippov solution.*

As a consequence from the relationships $\mathcal{K} = \mathcal{H}$ and $\mathcal{F} \subset \mathcal{K}$, we have the following lemma.

Lemma 3.1. *Suppose that the function $g : \mathbb{R}^n \to \mathbb{R}^n$ is Lebesgue measurable and locally bounded. Then $\mathcal{S} \subset \mathcal{H}$.*

However, we note that in general a Hermes solution is not a Sentis solution. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$ if $x > 0$, $f(x) = -1$ if $x < 0$ and $f(x) = 0$ if $x = 0$. Suppose that $a = 0$, $b = 1$ and $x_0 = 0$. Then $\mathcal{S} = \{t, -t\}$, what can be witnessed by [13]. On the other hand, $x(t) = 0$ is a Hermes solution.

4 Conclusions

The present work presents a study involving comparisons between generalized solutions for discontinuous differential equations. The main results on comparisons of generalized solutions are stated in Proposition 3.1, Proposition 3.2 and Lemma 3.1. Furthermore, in Theorem 3.2 it is proved that Hermes solutions also satisfy some properties which are analogous to the properties satisfied by the Euler solutions.

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