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The evolution by the curvature flow of the least diameter of a closed curve

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Resumo

The curvature flow of a curve was steadly studied in a series of papers by M. Gage, R. Hamilton [2], [3] and M. Grayson [4], that were published in the late 1980's. These works concern mainly the long time behavior of regular closed plane curves which deform in the direction of the curvature vectors. In general the curves shrink to a point, but it becomes more and more "round" which means that the curvature tends to a constant. In this paper we will see that in addition the minimum of the diameters of a curve decreases when it is undergone by the curvature flow action.

Palavras-chave

Curvature flow, curves in Euclidean space, Spaces of embeddings and immersions.

1 Introduction

Nonlinear heat equations have played a important role in differential geometry and topology over the last decades. Generally speaking, a geometric quantity or structure on a manifold is evolved in a canonical way towards an optimal one.

Examples are the harmonic map flow due to Eells and Sampson [8] which finds harmonic maps, that is local minima of the energy functional.

In 1996, Mullins [15] introduced the one-dimensional mean curvature flow, the curve shortening flow, in $\mathbb{R}^2$ and constructed examples of solitons for the flow.

A family of closed, embedded curves $\gamma(t)$ is given by $X(t,\cdot) : S^1 \to \mathbb{R}^2$ which evolve by the law of motion

$$\frac{\partial x}{\partial t} = k\upsilon \tag{1}$$

is called curve-shortening flow. Here υ is the inward pointing normal, k the curvature function of $\gamma(t)$, and X is short for $X(t,\cdot)$. We can express the equation also in terms of the arc length parameter $s = s(t,u)$ of the evolving curve. Using one of the definitions of curvature, the equation then reads as

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2}$$

This looks deceptively like a linear heat equation for X (actually a system of two equations in our case), however the non-linearity is hidden inside s which depends on products of the spatial derivatives of X. If, for instance, we express the evolving curve locally in terms of a function $h : (a,b) \times (0,T) \to \mathbb{R}$ with variables z and t then the equation

$$h_t = \frac{h_{zz}}{1 + h_z^2}$$

results. From equation (1), the evolution of any other geometric quantity on the curve can be computed [3]. The curvature, por instance, satisfies

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3$$

which is a so-called *reaction-diffusion equation*. If we only considered the diffusion part

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} \tag{2}$$

the curvature function $k(t)$ on $\gamma(t)$ would tend to a constant for $t \to \infty$. The curve would slowly turn into a circle.

The reaction part of the equation,

$$\frac{\partial x}{\partial t} = k^3,$$

with positive $k(0)$, has the explicit solution $k(t) = \dfrac{1}{\sqrt{\frac{1}{k(0)^2} - 2t}}$, so that the blow-up time is given by $T = \dfrac{1}{2k(0)^2}$.

In the equation(2) for the curvature function of $\gamma(t)$, these two effects are competing. It is an extremely difficult analytic problem to understand this interaction for a general initial curve. Fortunately, the geometric nature of the equation comes to the aid of the analysis(see [3], [4], [7]).

One of the most important features of the curve-shortening flow which it shares(after appropriate reformulation) with many other heat type equations is the comparison principle which says: Initially disjoint embedded closed curves stay disjoint during the curve-shortening flow. In fact, it follows from Grayson's work in 1987 [4], (see [7]for an easier proof) that the curve will stay smooth and embedded and will become convex before its extinction time. Then, by a result of Gage and Hamilton in 1984 [3], it will become asymptotically round in a smooth fashion. More precisely, after rescaling the evolving curve so as to keep for instance the enclosed area fixed(which results in a slightly different flow), it converges smoothly to a round circle. This is a consequence of the diffusion term: The diffusion cannot stop the formation of a singularity but it is strong enough to preserve embeddedness of the curve and produce a very “symmetric singularity".

It is worth pointing out that the curve-shortening flow [3] [4] and its higher dimensional analogue, the mean curvature flow [6], deform a curve(hypersurface) in the direction of its normal vector at every point, with speed equal to the curvature (mean curvature) at that point.

In 1982, Hamilton [13] introduced the Ricci flow which deforms an initial metric in the direction of its Ricci tensor. This flow tends to improve the manifold to one with locally homogeneous geometry. Hamilton used this to prove a number of topological classification results for manifolds with conditions on their curvature, such as for instance closed 3-manifolds with positive Ricci curvature and closed 4-manifolds with positive curvature operator(see [1] for description of his work and a complete list of references).

In 2003, Perelman [9], [10] and [11] completed Hamilton's Ricci flow programme [5] which had the aim of settling Thurston's geometrization conjecture for closed 3-manifolds [14]. This conjecture had predictec such manifolds to be decomposable into pieces with locally homogeneous geometry.

2 The curve shortening flow

The problem of curve shortening or curvature flow is a particular case which one consist in deformer a immersion between Riemannian manifolds by the heat equation. in other words, if $F : \mathbf{M} \to \mathbf{N}$ is a smooth isometric immersion, the Laplacian of F is defined intrinsically as a section of pull-back of the tangent bundle of M, and given by $\Delta F = kN$, where k is the mean curvature and N is the normal unit vector. The immersion F can be deformed by the heat equation $\frac{\partial F}{\partial t} = \Delta F$ or $\frac{\partial F}{\partial t} = kN$. In [13] is proved that the solution always exists for a short time, is unique and smooth and it does not depend of the parametrization chosen.

In a variational sense, that is, in the spitrit of the Calculus of variations, we can describe the shortening flow problem in the following way: The space $\mathcal{M}$ of all immersed submanifolds $\mathbf{M}$ of $\mathbf{N}$ has the structure of an infinite- dimensional manifold modeled on a Fréchet space [12], the tangent space $T_{\mathbf{M}}\mathcal{M}$ to $\mathbf{M}$ at $\mathcal{M}$ is naturally identified with the space $C^{\infty}(\mathbf{M})$ of functions f on $\mathbf{M}$ where the variation in $\mathbf{M}$ is given by a small(infinitesimally) moving in the normal direction. The volume $V(\mathbf{M})$ of $\mathbf{M}$ gives a function on $\mathcal{M}$ whose derivative in the direction of a normal variation is $DV(\mathbf{M}) = -\int_{\mathbf{M}} fk$. It follows that the heat equation $\frac{\partial F}{\partial t} = -kN$ describes the gradient flow for the Morse function V. In the particular case when $\mathbf{M}$ is a convex curve embedded in the plane, the curvature flow shrinks $\mathbf{M}$ to a point. The curve remains convex and becomes circular as it shrinks.

We consider family of regular (C^r, $r \geq 2$), simple, closed curves in $\mathbb{R}^2$, with euclidean metric, $X(t, u)$ that satisfies the curve shortening flow:

$$\frac{\partial X}{\partial t}(t, u) = k(t, u)N(t, u)$$

Where $k(t, u)$ is the curvature and $N(t, u)$ is the normal (unitary) vector pointing inward. Observe that u is not the arc-length parameter.

Some of the main results about this flow are listed below:

Theorem 1 (Gage-Hamilton). *The curve shortening flow preserves convexity and shrinks any closed simple convex curve to a point.*

Theorem 2 (Grayson). *Starting with any closed curve it becomes convex before it shrinks to a point.*

We refer the reader to [3] and [4] for their proofs and consequences of these results.

Just to fix some notation that will be used in sequence we shall prove a very well known fact concerning the evolution equation for the area, $A(t)$, enclosed by a simple closed curve(for more detail see [3] pag. 75 lemma 3.1.6). Here we are using the standard language and also the notation of classical theory of curves.

Proposition 1. *Let $A(t)$ be the area enclosed by the curve $\gamma(t) = X(t, .)$ and $A(t) = \frac{1}{2}\int_{\gamma(t)} xdy - ydx$. Then $A'(t) = 2\pi$.*

Proof. It is a well kown fact from the theory curves that we can interprete the area as being

$$A = \frac{1}{2}\int_0^{2\pi} -ydx + xdy = \frac{1}{2}\int_0^{2\pi}\left(-y\frac{dx}{du} + x\frac{dy}{du}\right)du = -\frac{1}{2}\int_0^{2\pi} < X, vN > du \quad (3)$$

where $v = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2}$, thus computing the derivative of $A(t)$ with respect to t we obtain

$$\begin{aligned}\frac{dA}{dt} &= -\frac{1}{2}\int_0^{2\pi} < \frac{\partial X}{\partial t}, vN > + < X, \frac{\partial v}{\partial t}N > + < X, -v\frac{\partial k}{\partial s}T > du \\ &= -\frac{1}{2}\int_0^{2\pi} kv - < X, -k^2 vN > -v\frac{\partial k}{\partial s} < X, T > du \qquad (4) \\ &= -\frac{1}{2}\int_0^{2\pi} kv - < X, -k^2 vN > -\frac{\partial k}{\partial u} < X, T > du\end{aligned}$$

By integrating the last term follows that:

$$\begin{aligned}\frac{dA}{dt} = & -\frac{1}{2}\int_0^{2\pi} kv - k^2 v < X, N > +k < \frac{\partial X}{\partial u}, T > +k^2 < X, N > du \\ &= -\int_0^{2\pi} kvdv = -\int_0^{L} kds \qquad (5) \\ &= -\int_0^{2\pi} kd\theta = 2\pi\end{aligned}$$

and the result follows. □

From the previous lemma we get two consequences:

The first is related to the existence time τ of a solution. We conclude that it is given by $\tau = \dfrac{A(0)}{2\pi}$

The second is that there is a natural normalization for the flow. It is obtained by taking a homothety $Y(t,u) = \mu(t)X(t,u)$ such that the area of the region enclosed by the curve $Y(t,.)$ is constant equal to π. For that, it is enough to define $\mu(t)$ as being $\mu(t) = \dfrac{1}{\sqrt{\tau - 2t}}$.

Moreover, there is also a time normalization given by $\bar{t} = \frac{1}{2}ln[\frac{\tau}{\tau - t}]$, so that $t \to \tau$ implies $\bar{t} \to \infty$.

For the normalized flow, the area enclosed by the curve is constant and equals to π and Gage and Hamilton prove that the family converges uniformly to a circle of radius 1.

3 The shortening of the diameter

We will use the notation (||.||) for the Euclidean distance. The next lemma will be used to prove our main result.

Lemma 1. *The function lenght cord* $L(t,u_1,u_2) = ||X(t,u_2) - X(t,u_2)||$ *evolves by the curvature flow and satisfies the following equation:*

$$\frac{\partial L}{\partial t} = \Delta L + \frac{2}{L}||\nabla L||^2 - \frac{2}{L} \tag{6}$$

Proof. Instead of computing $\dfrac{\partial L}{\partial t}$ directly we will proceed as follows

$$\frac{\partial}{\partial t}L^2 = 2L\frac{\partial L}{\partial t} = 2 < k(u_2)N(u_2) - k(u_1)N(u_1), X(t,u_2) - X(t,u_1) > .$$

From the previous equation we obtain

$$\frac{\partial L}{\partial t} = \frac{1}{L} < k(u_2)N(u_2) - k(u_1)N(u_1), X(t,u_2) - X(t,u_1) > . \tag{7}$$

Moreover, if s is the arclenght variable of $X(t, \dot{})$, we obtain

$$\frac{\partial}{\partial s_2}L^2 = 2L\frac{\partial L}{\partial s_2} = 2 < T(s_2), X(t,s_2) - X(t,s_1) >$$

which implies that

$$2(\frac{\partial L}{\partial s_2})^2 + 2L\frac{\partial^2 L}{\partial s_2^2} = 2L\frac{\partial L}{\partial s_2} = 2\frac{\partial}{\partial s_2} < T(s_2), X(t,s_2) - X(t,s_1) >$$

and therefore

$$\frac{\partial^2 L}{\partial s_2^2} = -\frac{1}{L}(\frac{\partial L}{\partial s_2})^2 + \frac{1}{L} - \frac{1}{L} < k(s_2)N(s_2), X(t,s_2) - X(t,s_1) > . \tag{8}$$

In the same way, we also obtain:

$$\frac{\partial^2 L}{\partial s_1^2} = -\frac{1}{L}(\frac{\partial L}{\partial s_1})^2 + \frac{1}{L} - \frac{1}{L} < k(s_1)N(s_1), X(t,s_2) - X(t,s_1) > \qquad (9)$$

The result follows from (7), (8) and (9). □

Now we define the diameter of a curve X by being the lenght of the vector $X(s_1) - X(s_2)$ which is orthogonal to their respectives unit tangent vectors $T(s_1),\ T(s_2)$, e.g, $X(s_1) - X(s_2) \perp \{T(s_1), T(s_2)\}$. Now we are in condition to prove our main result that is the following:

Theorem 3. *Let $L_{\min}(t)$ be the minimum of the lenght of all diameters of the curve $X(t,.)$ Let us suppose that the minimum of the diameters is nondegenerate point, then $L_{\min}(t)$ is a non-decreasing function with respect to t, when $X(t,\cdot)$ evolves by the curvature flow.*

Proof. Let us take $\epsilon > 0$ such that $L_{\min}(0) > \epsilon > 0$. The existence of this number is assured by the fact that the initial curve, and consequently all the others, are embedded curves in $\mathbb{R}^2$. Let us suppose that $L_{\min}(t) = L_{\min}(0) - \epsilon$ for some t.

Let $t_0 = \inf\{t | L_{\min}(t) = L_{\min}(0) - \epsilon\}$. The continuity of L assures that this minimum is reached at some point $(t_0, \hat{u}_1, \hat{u}_2)$. Consequently, at this point, we have

$$\frac{\partial L}{\partial t} \leq 0,\ \frac{\partial L}{\partial s_1} = \frac{\partial L}{\partial s_2} = 0$$

and

$$\frac{\partial^2 L}{\partial s_1^2}\frac{\partial^2 L}{\partial s_2^2} - (\frac{\partial^2 L}{\partial s_1 \partial s_2})^2 > 0$$

(by the hypothesis we are assuming that the critical points are non-degenerate).

Other hand, since $\frac{\partial^2 L}{\partial s_1 \partial s_2} = \frac{1}{L} < T(u_1), T(u_2) >= -\frac{1}{L}$ and $\Delta L = \frac{\partial^2 L}{\partial s_1^2} + \frac{\partial^2 L}{\partial s_2^2} \geq 2\sqrt{\frac{\partial^2 L}{\partial s_1^2}\frac{\partial^2 L}{\partial s_2^2}} > 2|\frac{\partial^2 L}{\partial s_1^2}\frac{\partial^2 L}{\partial s_2^2}| = \frac{2}{L}$.

Then

$$0 \geq \frac{\partial L}{\partial t} = \Delta L + \frac{2}{L}||\nabla L||^2 - \frac{2}{L} > \frac{2}{L} - \frac{2}{L} = 0$$

Which is a contradiction. Then the result follows.

□

4 Final remark

I think it is possible extend the same result for compact surfaces, e.g, the minimum of diameter is a non-decreasing function when the metric evolves by Ricci flow. For this we will shoud do a analysis of the evolution of the metrics by Ricci flow. This will be donne in a future work.

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