RMAT
ISSN: 2237-8103
REVISTA DE MATEMÁTICA DA UFOP

Volume 1, Número 1, 202x
<https://doi.org/10.5281/zenodo.11062161>
Recebido: 01/04/2024
Publicado: 24/04/2024

ON THE REPUNIT SEQUENCE AT NEGATIVE INDICES

Eudes Antonio Costa
<eudes@uft.edu.br>
Universidade Federal do Tocantis, UFT, Arraias, To, Brasil
<https://orcid.org/0000-0001-6684-9961>

Douglas Catulio dos Santos
<catuliodouglas4@gmail.com>
Secretaria de Educação do Estado da Bahia, Barreiras, BA, Brasil
<https://orcid.org/0000-0002-5221-6087>

Francival Santos Monteiro
<francival.monteiro@uft.edu.br>
Universidade Federal do Tocantis, UFT, Arraias, To, Brasil
<https://orcid.org/0009-0003-0457-7667>

Vitor Manoel Alves de Souza
<souza.alves@uft.edu.br>
Universidade Federal do Tocantis, UFT, Arraias, To, Brasil
<https://orcid.org/0009-0008-5017-7089>

Abstract
In this work we will present an extension of the repunit sequence related to repunit numbers with a negative subscripts. Our main objective is to establish properties of this new sequence, as well as the Binet formula, the generating functions and the classical identities. The identities of Catalan, Cassini and d'Ocagne related to a sequence of numbers are important because they describe an elegant relationship between the elements of the sequence.

Keywords: *Negative Repunit Sequence. Generating Functions. Identities*

1. INTRODUCTION

The *repunit* sequence is formed by numbers, elements of the sequence, which are written in the decimal system as the repetition of the unit, represented by $\{r_n\}_{n\geq 1} = \{1,\ 11,\ 111, \ldots,\ r_n, \ldots\}$, the sequence $A002275$ in OEIS (SLOANE et al., 2024). See that, for all $n \geq 1$, we have that

$$r_n = 10 \cdot r_{n-1} + 1, \text{ with } r_0 = 0\ . \tag{1}$$

Where r_n denotes the n-th repunit, and for convenience we use $r_0 = 0$. In (SANTOS; COSTA, 2023), for $n \geq 1$, introduced another recurrence relation, related to the initial recurrence (1). Making $r_{n+1} - r_n$, and so the repunit sequence is also determined by the homogeneous relation

$$r_{n+1} = 11r_n - 10r_{n-1}\ , \tag{2}$$

with initial condition $r_0 = 0$ and $r_1 = 1$.

The repunit sequence, recurrence (2), is a special case of Horadam sequence, where Horadam sequence is defined by

$$W_{n+1} = pW_n + qW_n;\ (n \geq 0)$$

where $W_0 = a; W_1 = b$ for all $a, b, p, q \in \mathbb{R}$. So if we take $p = 11; q = -10; a = 0$; and $b = 1$; then the Horadam sequence is reduced to repunit sequence. More general results found see (CERDA, 2012; TAŞTAN; YILMAZ; ÖZKAN, 2022) .

An approach to the study of numerical sequences is through matrix representation, and in specialized literature, we find several works that relate different types of sequences and matrices, of which we can mention (COSTA; SANTOS, 2022b; FALCON, 2013; KALMAN, 1982; KILIC, 2007; KILIC; OEMUER; ULUTAŞ, 2009; KILIC; TASCI; HAUKKANEN, 2010; KILIÇ; STANICA, 2011). Specifically, a study on the repunit sequence and its matrix representation can be found in (SANTOS; COSTA, 2024). Another approach to studying numerical sequences is to extend the sequence to integer numbers, that is, also to negative indices, of which we can mention (DASDEMIR, 2019; HALICI; AKYÜZ, 2016; HORADAM, 1982; MANGUEIRA et al., 2021; SOYKAN, 2021).

The repunit sequence $\{r_n\}_{n\geq 0}$ can be extended to negative subscripts by the following definition.

Definition 1. *Let $n \geq 1$, then the negative index n-th repunit numbers is defined as*

$$r_{-n} = -\frac{r_n}{10^n}\ .$$

It follows from the definition that repunit sequence with negative index is the set of elements given by

$$\{r_{-n}\}_{n\geq 1} = \left\{-\frac{1}{10}, -\frac{11}{10^2}, -\frac{111}{10^3}, \dots, \right\} = \{-0,1;\ -0,11;\ -0,111, \dots, \}\ .$$

The first few repunit numbers with negative subscript are given in the following Table 1, with $-8 \leq n \leq -1$:

n	-8	-7	-6	-5	-4	-3	-2	-1
r_n	- 0,11111111	-0,1111111	-0,111111	-0,11111	-0,1111	-0,111	-0,11	-0,1

TABLE 1.
Repunit numbers at negative index

Observation of Table 1 gives rise to,

$$\frac{11}{10}r_{-2} - \frac{1}{10}r_{-1} = -\frac{11}{10}\cdot\frac{11}{100} + \frac{1}{10}\cdot\frac{1}{10} = \frac{121}{1000} + \frac{1}{100} = -\frac{111}{1000} = -0,111 = r_{-3}\ ,$$

as well as

$$\frac{11}{10}r_{-3} - \frac{1}{10}r_{-2} = -\frac{11}{10}\cdot\frac{111}{1000} + \frac{1}{10}\cdot\frac{11}{100} = -\frac{1221}{10000} + \frac{11}{1000} = -\frac{1111}{10000} = -0,1111 = r_{-4}\ .$$

The following results demonstrates this fact.

Proposition 2. *The repunit sequence with negative index satisfies the recurrence relation*

$$r_{-(n+1)} = \frac{11}{10}r_{-n} - \frac{1}{10}r_{-(n-1)} \text{ with } r_{-1} = -0,1 \text{ and } r_{-2} = -0,11\,; \tag{3}$$

for $n = 1, 2, 3, \ \ldots$.

Proof. See that

$$\begin{aligned} \frac{11}{10}r_{-n} - \frac{1}{10}r_{-(n-1)} &= -\frac{11}{10}\cdot\frac{r_n}{10^n} + \frac{1}{10}\cdot\frac{r_{n-1}}{10^{n-1}} \\ &= \frac{-11r_n + 10r_{n-1}}{10^{n+1}} \\ &= -\frac{r_{n+1}}{10^{n+1}}\,. \end{aligned}$$

□

Therefore, the recurrence equation (3) is given by a recurrence relation of order 2, as well as recurrence equation (2). So, recurrence (2) holds for all integer n.

Here, we present and study a generalization of the terms of the repunit sequence to negative indices, describing a new numerical sequence denoted by r_{-n}. The Proposition 6, in Section 2, provides the Binet Formula for negative indices, characterizing the terms of r_{-n}. In Section 3, we demonstrate the classical Catalan, Cassini, and D'Ocagne identities for r_{-n}. In Section 4, we present results on partial sums of terms of the repunit sequence with n integers. Finally, in Section 5, taking the quotients (divisions) of each term by its predecessor, we determine the limit of this numerical sequence whose general term is r_{n+1}/r_n, that is, it is a sequence limited.

2. LINEAR RECURRENCE AND BINET'S FORMULA

See that Equation (3) is a linear difference equation of order 2. Therefore, to determine a solution to the difference equation we will present the following auxiliary results (Lemmas), whose demonstrations can be consulted in (CARVALHO; MORGADO, 2015; ROSEN, 2007) .

Lemma 3. *The linear difference equation, given by*

$$x_{n+2} + px_{n+1} + qx_n = 0$$

with $x_1 = a_1$, $x_2 = a_2$, *and* a_1, $a_2 \in \mathbb{R}$ *and* $n \in \mathbb{N}$, *has a single solution.*

Lemma 4. *If the equation* $r^2 + pr + q = 0$ *has distinct roots* r_1 *and* r_2, *the sequences* $a_n = c_1(r_1)^n + c_2(r_2)^n$, *where* $n \in \mathbb{N}$, *and* c_1, $c_2 \in \mathbb{R}$, *are solutions of*

$$x_{n+2} + px_{n+1} + qx_n = 0, \text{ para } n \in \mathbb{N}, n \geq 1\,.$$

In the decimal system, according to (BEILER, 1964; COSTA; SANTOS, 2022a; RIBENBOIM, 2004; TARASOV, 2007) the equation

$$r_n = \frac{10^n - 1}{9} \tag{4}$$

presents Binet's formula for repunit numbers.

See that the recurrence $r_{-(n+1)} = \frac{11}{10} r_{-n} - \frac{1}{10} r_{-(n-1)}$ has characteristic equation given by

$$x^2 - \frac{11}{10}x + \frac{1}{10} = 0\,, \tag{5}$$

whose roots are $x_1 = \frac{1}{10}$ and $x_2 = 1$. Let's determine the real constants c_1 and c_2, considering that $r_{-1} = 0,1$ and $r_{-2} = 0,11$, and we obtain the linear system,

$$\begin{cases} -0,1 = & c_1\left(\frac{1}{10}\right) + & c_2 \\ -0,11 = & c_1\left(\frac{1}{100}\right) + & c_2 \end{cases}$$

Solving the system we find $c_1 = \frac{-1}{9}$ and $c_2 = \frac{1}{9}$. Then, under the previous discussion, we can provide the Binet formula, as follows.

Proposition 5 (Binet's formula). *For all $n \in \mathbb{N}$, we have*

$$r_{-n} = -\frac{10^n - 1}{9 \cdot 10^n}\,. \tag{6}$$

Proof. We have that a general solution to Equation (5) is of the form $r_{-n} = c_1\left(\frac{1}{10}\right)^n + c_2(1)^n$. Then, we obtain

$$\begin{aligned} r_{-n} &= c_1\left(\frac{1}{10}\right)^n + c_2(1)^n \\ &= \frac{-1}{9}\left(\frac{1}{10}\right)^n + \frac{1}{9}(1)^n \\ &= -\frac{10^n - 1}{9 \cdot 10^n}\,. \end{aligned}$$

□

The function, according (SPREAFICO; CRAVEIRO; RACHIDI, 2022),

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \tag{7}$$

is known as the generating function for the sequence $\{a_0, a_1, a_2, \ldots\}$. Generating functions provide an interesting tool in solving second-order linear recurrences, as expressed in the Horadan and Lucas sequences. Therefore, our next result presents a generating function for the negative index repunits.

Proposition 6. *The generating function for the repunits r_{-n}, denoted by $gr_{-n}(x)$, is:*

$$gr_{-n}(x) = \frac{x}{10 - 11x + 1x^2}\,.$$

Proof. According to Equation (7), the generating function for the repunit sequence is $gr_{-n}(x) = \sum_{n=0}^{\infty} r_{-n}x^n$, then using the equations $-\frac{11x}{10}gr_{-n}$ and $\frac{1x^2}{10}gr_{-n}$, we obtain

$$\begin{aligned} gr_{-n}(x) =& r_0 + r_{-1}x + r_{-2}x^2 + \ldots + r_{-n}x^n + \ldots \\ -\frac{11x}{10}gr_{-n}(x) =& -\frac{11x}{10}r_0 - \frac{11x}{10}r_{-1}x - \frac{11x}{10}r_{-2}x^2 - \ldots - \frac{11x}{10}r_{-n}x^n - \ldots \\ \frac{1x^2}{10}gr_{-n}(x) =& \frac{1x^2}{10}r_0 + \frac{1x^2}{10}r_{-1}x + \frac{1x^2}{10}r_{-2}x^2 + \ldots + \frac{1x^2}{10}r_{-n}x^n + \ldots \end{aligned}$$

When we add to both sides, we have:

$$\begin{aligned} &\left(1 - \frac{11x}{10} + \frac{1}{10}x^2\right) gr_{-n}(x) \\ =& r_0 + (r_{-1} - \frac{11}{10}r_0)x + (r_{-2} - \frac{11}{10}r_{-1} + \frac{1}{10}r_0)x^2 + \\ &+ (r_{-3} - \frac{11}{10}r_{-2} + \frac{1}{10}r_{-1})x^3 + \ldots + (r_{-n} - \frac{11}{10}r_{-n+1} + \frac{1}{10}r_{-n+2})x^n \ldots \\ \overset{(5)}{=}& \frac{1}{10}x + 0 \cdot x^2 + 0 \cdot x^3 + \ldots + 0 \cdot x^n + \ldots \end{aligned}$$

Where the result follows without much effort. □

According (SPREAFICO; CRAVEIRO; RACHIDI, 2022) the exponential generating function $er_{-n}(x)$ of a sequence $\{a_n\}_{n\geq 0}$ is a power series of the form

$$ea_n = a_0 + a_1x + \frac{a_2x^2}{2!} + \ldots + \frac{a_nx^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{a_nx^n}{n!} .$$

In the next result we consider the Binet equation (6), and obtain the exponential generating function for the repunit sequence $\{r_{-n}\}_{n\geq 0}$.

Theorem 7. *For all $n \geq 0$ the exponential generating function for the repunit sequence $\{r_{-n}\}_{n\geq 0}$ is*

$$er_{-n} = \frac{e^{\frac{t}{10}} - e^t}{9} .$$

Proof. The exponential generating function for the repunit numbers is $\sum_{n=0}^{\infty} \frac{r_{-n}t^n}{n!}$. Using Equation (6), we obtain that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{r_{-n}t^n}{n!} &= -\sum_{n=0}^{\infty} \frac{10^n - 1}{9 \cdot 10^n} \cdot \frac{t^n}{n!} \\ &= \frac{1}{9}\left(\sum_{n=0}^{\infty} \frac{\left(\frac{t}{10}\right)^n}{n!} - \sum_{n=0}^{\infty} \frac{t^n}{n!}\right) \\ &= \frac{1}{9}\left(e^{\frac{t}{10}} - e^t\right) . \end{aligned}$$

□

3. SOME IDENTITIES

In the literature for a numerical sequence U_n, the identities $U_m^2 - U_{m+n}U_{m-n} = X$, $U_m^2 - U_{m+1}U_{m-1} = Y$ and $U_mU_{n+1} - U_{m+1}U_n = Z$, are known, respectively, as the Catalan Identity, the Cassini Identity and the d'Ocagne Identity, where X , Y and Z are integers. Santos and Costa (SANTOS; COSTA, 2023) in turn, exhibit the identities for the positive indices of the repunit sequence.

Now we present too for negative indices of some classical identities of the repunit sequence. So, in this section, we provide the Catalan, Cassini, and d'Ocagne identities related to the negative sequence repunit.

Proposition 8. *[Catalan's Identity] Let m, n be any natural. For $m \geq n$ we have*

$$(r_{-m})^2 - r_{-(m-n)}r_{-(m+n)} = \frac{(r_n)^2}{10^{(m-n)}} .$$

Proof. By Equation (6), we have that

$$\begin{aligned}(r_{-m})^2 - r_{-(m-n)}r_{-(m+n)} &= \left(-\frac{10^m - 1}{9 \cdot 10^m}\right)^2 - \left(-\frac{10^{m-n} - 1}{9 \cdot 10^{m-n}}\right)\left(-\frac{10^{m+n} - 1}{9 \cdot 10^{m+n}}\right) \\ &= \frac{10^{2m} - 2 \cdot 10^m + 1 - 10^{2m} + 10^{m-n} + 10^{m+n} - 1}{81 \cdot 10^{2m}} \\ &= \frac{10^{(m-n)} + 10^{(m+n)} - 2 \cdot 10^m}{81 \cdot 10^{2m}} \\ &= \frac{10^{m-n}(10^{2n} - 2 \cdot 10^n + 1)}{81 \cdot 10^{2m}} \\ &= \frac{10^{m-n}}{10^{2m}}\left(\frac{10^n - 1}{9}\right)^2 ,\end{aligned}$$

the result is derived from the combined application of the properties of powers and the Proposition 6. ☐

Setting $n = 1$ in Proposition 8, we obtain the following identity for negative indices:

Proposition 9 (Cassini's Identity)**.** *For all $m \geq 1$, we have $(r_{-m})^2 - r_{-(m-1)}r_{-(m+1)} = 10^{-(m-1)}$.*

Similar to Proposition 8 we have:

Proposition 10 (D'ocagne's Identity)**.** *Let $m,\ n$ be any natural. For $m \geq n$ we have*

$$r_{-m}r_{-(n+1)} - r_{-(m+1)}r_{-n} = \frac{r_{m-n}}{10^{m+1}} .$$

Proof. Using a Equation (6) again we obtain that

$$\begin{aligned}
& r_{-m} \cdot r_{-(n+1)} - r_{-(m+1)} \cdot r_{-n} \\
&= \left(-\frac{10^m - 1}{9 \cdot 10^m}\right) \cdot \left(-\frac{10^{n+1} - 1}{9 \cdot 10^{n+1}}\right) - \left(-\frac{10^{m+1} - 1}{9 \cdot 10^{m+1}}\right)\left(-\frac{10^n - 1}{9 \cdot 10^n}\right) \\
&= \left(\frac{10^{m+n+1} - 10^m - 10^{n+1} + 1}{9^2 \cdot 10^{m+n+1}}\right) - \left(\frac{10^{m+n+1} + 10^{m+1} - 10^n + 1}{9^2 \cdot 10^{m+n+1}}\right) \\
&= \left(\frac{-10^m - 10^{n+1} + 10^{m+1} + 10^n}{9^2 \cdot 10^{m+n+1}}\right) \\
&= \frac{10^{m+1} - 10^m + 10^n - 10^{n+1}}{9^2 \cdot 10^{m+n+1}} \\
&= \frac{10^m(10-1) - 10^n(10-1)}{9^2 \cdot 10^{m+n+1}} \\
&= \frac{9(10^m - 10^n)}{9^2 \cdot 10^{m+n+1}} \\
&= \frac{10^m - 10^n}{9 \cdot 10^{m+n+1}} .
\end{aligned}$$

Since $m \geq n$, we have

$$\begin{aligned}
r_{-m} \cdot r_{-(n+1)} - r_{-(m+1)} \cdot r_{-n} &= \frac{10^n(10^{m-n} - 1)}{9 \cdot 10^{m+n+1}} \\
&= \frac{10^{m-n} - 1}{9 \cdot 10^{m+1}} ,
\end{aligned}$$

and we have the validity of the result. ☐

4. SUM FORMULAS

Considering the sequence of partial sums $S_n = r_1 + r_2 + \cdots + r_n$, for $n \geq 1$, where $\{r_n\}_{n\geq 1}$ is the repunit sequence. We have that:

Proposition 11. *Let $(r_n)_{n\geq 1}$ be the repunit sequence, then*

$$\begin{aligned}
\text{(a)} \sum_{k=0}^{n} r_k &= \frac{10r_n - n}{9} , \\
\text{(b)} \sum_{k=0}^{n} r_{2k} &= \frac{10^2 r_n - nr_2}{99} , \\
\text{(c)} \sum_{k=0}^{n} r_{2k+1} &= \frac{r_{2n+3} - (n+1)r_2}{99} .
\end{aligned}$$

Proof. (a) From of Proposition 9 in (SANTOS; COSTA, 2023) is valid that $\sum_{k=0}^{n} r_k = \frac{10(10^n - 1) - 9n}{81}$.

So

$$\sum_{k=0}^{n} r_k = 10\frac{10^n - 1}{9 \cdot 9} - \frac{9n}{9 \cdot 9} = \frac{10r_n - n}{9} .$$

(b) See that $S_{2n} = r_0 + r_2 + \cdots + r_n$. Now, it follows from Equation (4) that

$$\begin{aligned} S_{2n} &= \frac{10^2-1}{9} + \frac{10^4-1}{9} + \cdots + \frac{10^{2n}-1}{9} \\ &= \frac{(10^2+10^4+\cdots+10^{2n})-n}{9} \\ &= \frac{\frac{10^2(10^{2n}-1)}{99}-n}{9} = \frac{10^2(10^{2n}-1)-99n}{891} \\ &= \frac{10^2 r_n - 11n}{99} . \end{aligned}$$

(c) In a similar way

$$\begin{aligned} S_{2n+1} &= r_1 + r_3 + \cdots + r_{2n+1} \\ &= \frac{10-1}{9} + \frac{10^3-1}{9} + \cdots + \frac{10^{2n+1}-1}{9} \\ &= \frac{(10+10^3+\cdots+10^{2n+1})-(n+1)}{9} \\ &= \frac{\frac{10(10^{2n+2}-1)}{99}-(n+1)}{9} = \frac{(10^{2n+3}-1)-99n-108}{891} \\ &= \frac{r_{2n+3}-11n-12}{99} . \end{aligned}$$

□

Now, let S_{-n} be the sequence formed by partial sums $S_{-n} = r_{-1} + r_{-2} + r_{-3} + \ldots + r_{-n}$, with $n \geq 1$, where $\{r_{-n}\}_{n\geq 1}$ represents the terms of negative order in the repunit sequence.

Proposition 12. *Let $\{r_{-n}\}_{n\geq 1}$ be the repunit negative sequence, then*

$$\begin{aligned} &\textit{(a)}\ \sum_{k=0}^{n} r_{-k} &&= -\frac{n-r_{-n}}{9} , \\ &\textit{(b)}\ \sum_{k=0}^{n} r_{-2k} &&= -\frac{nr_2-r_{-2n}}{99} , \\ &\textit{(c)}\ \sum_{2k=0}^{n} r_{-(2k+1)} &&= -\frac{(n+1)r_2-r_{-(2n+1)}}{99} . \end{aligned}$$

Proof. (a) We have that

$$\sum_{k=0}^{n} r_{-k} = r_{-1} + r_{-2} + r_{-3} + \ldots + r_{-n} .$$

Now, it follows from Equation 6 that

$$\begin{aligned}
S_{-n} &= r_{-1}+r_{-2}+r_{-3}+\ldots+r_{-n}\\
&= -\left(\frac{10-1}{10\cdot 9}+\frac{10^2-1}{10^2\cdot 9}+\frac{10^3-1}{10^3\cdot 9}+\ldots+\frac{10^n-1}{10^n\cdot 9}\right)\\
&= -\frac{10^{n-1}(10-1)+10^{n-2}(10^2-1)+\ldots+10(10^{n-1}-1)+10^n-1}{10^n\cdot 9}\\
&= -\frac{10^n\cdot n-(10^{n-1}+10^{n-2}+10^{n-3}+\ldots+10+1)}{10^n\cdot 9}\\
&= \frac{-n10^n+\frac{10^n-1}{10-1}}{10^n\cdot 9} = \frac{-9n10^n+(10^n-1)}{10^n\cdot 9^2}\\
&= -\frac{9n10^n}{10^n\cdot 9^2}+\frac{10^n-1}{10^n\cdot 9^2} = -\frac{n}{9}+\frac{r_{-n}}{9},
\end{aligned}$$

(b) In a similar way

$$\begin{aligned}
S_{-2k} &= r_{-2}+\ldots+r_{-2n}\\
&= -\left(\frac{10^2-1}{10^2\cdot 9}+\frac{10^4-1}{10^4\cdot 9}+\frac{10^6-1}{10^6\cdot 9}+\ldots+\frac{10^{2n}-1}{10^{2n}\cdot 9}\right)\\
&= -\frac{10^{2n-2}(10^2-1)-10^{2n-4}(10^4-1)+\ldots+10^{2n}-1}{10^{2n}\cdot 9}\\
&= \frac{-10^{2n}\cdot n+(10^{2n-2}+10^{2n-4}+10^{2n-6}+\ldots+10^2+1)}{10^{2n}\cdot 9}\\
&= \frac{-10^{2n}\cdot n+\frac{10^{2n}-1}{99}}{10^{2n}\cdot 9} = \frac{-10^{2n}\cdot n}{10^{2n}\cdot 9}+\frac{10^{2n}-1}{10^{2n}\cdot 9\cdot 99}\\
&= -\frac{n}{9}+\frac{10^{2n}-1}{10^{2n}\cdot 891} = -\frac{n\cdot r_2-r_{-2n}}{99}.
\end{aligned}$$

(c) To the same effect

$$\begin{aligned}
S_{-(2k+1)} &= r_{-1}+r_{-3}+r_{-5}+\ldots+r_{-(2n+1)}\\
&= -\left(\frac{10-1}{10\cdot 9}+\frac{10^3-1}{10^3\cdot 9}+\frac{10^5-1}{10^5\cdot 9}+\ldots+\frac{10^{2n+1}-1}{10^{2n+1}\cdot 9}\right)\\
&= -\frac{10^{2n}(10-1)+10^{2n-2}(10^3-1)+10^{2n-4}(10^5-1)+\ldots+10^{2n+1}-1}{10^{2n+1}\cdot 9}\\
&= -\frac{10^{2n+1}(n+1)-(10^{2n}+10^{2n-2}+\ldots+10^2+1)}{10^{2n+1}\cdot 9}\\
&= -\frac{10^{2n+1}(n+1)-\frac{10^{2n+1}-1}{99}}{10^{2n+1}\cdot 9}\\
&= -\frac{n+1}{9}+\frac{10^{2n+1}-1}{10^{2n+1}\cdot 9\cdot 99}\\
&= -\frac{(n+1)r_2-r_{-(2n+1)}}{99}.
\end{aligned}$$

and we have the result. □

The following example illustrates the result demonstrated in Proposition 12.

Example 13. *Note that for* $n = 4$*, the sum of the* $4-th$ *of* $\{r_{-n}\}_{n\geq 1}$ *is given by* $S_{-4} = r_{-1}+r_{-2}+r_{-3}+r_{-4} = -\frac{4-0,1111}{9} = -0,4321$*, while if* $n = 8$*, we obtain* $S_{-8} = r_{-1}+r_{-2}+r_{-3}+r_{-4}+r_{-5}+r_{-6}+r_{-7}+r_{-8} = -\frac{8-0,11111111}{9} = -0,8764321.$

5. LIMIT

Once more, using the Binet formulas (4) and (6) we obtain another property of the repunit sequences $\{r_n\}_{n\in\mathbb{Z}}$ which is stated in the following proposition.

Proposition 14. *If* r_n *are the* n*-th terms of repunit sequence, , then*

$$\lim_{n\to\infty}\frac{r_{n+1}}{r_n} = 10\,, \tag{8}$$

and

$$\lim_{n\to-\infty}\frac{r_{-(n+1)}}{r_{-n}} = \frac{1}{10}\,. \tag{9}$$

Proof. We have that

$$\lim_{n\to\infty}\frac{r_{n+1}}{r_n} = \lim_{n\to\infty}\frac{10^{n+1}-1}{9}\cdot\frac{9}{10^n-1} = \lim_{n\to\infty}\frac{10-\frac{1}{10^n}}{1-\frac{1}{10^n}} = 10\,,$$

since $\lim_{n\to\infty}\frac{1}{10^n} = 0$. And

$$\lim_{n\to-\infty}\frac{r_{-(n+1)}}{r_{-n}} = \lim_{n\to-\infty}\frac{10^{n+1}-1}{9\cdot 10^{n+1}}\cdot\frac{9\cdot 10^n}{10^n-1} = \lim_{n\to\infty}\frac{10^{n+1}-1}{10^{n+1}-10} = \frac{1}{10}\,,$$

since $\lim_{n\to-\infty} 10^n = 0.$ □

In what follows, we can easily show the next result using basic tools of calculus of limits, (8) and (9).

Corollary 15. *If* r_n *are the* n*-th terms of repunit sequence, , then*

$$\lim_{n\to\infty}\frac{r_n}{r_{n+1}} = \frac{1}{10}\,,$$

and

$$\lim_{n\to-\infty}\frac{r_{-n}}{r_{-(n+1)}} = 10\,.$$

6. CONSIDERATIONS

In this work we discuss some results about the repunit sequence with negative indices. We present a generalization of the repunit sequence, a version of Binet's Formula for negative indices, and the generating function for r_{-n}, thus establishing the continuity of the function over the entire set of integers. In addition, we characterize the classical identities of recurrent sequences for the entire repunit sequence: the Catalan, Cassini, and D'Ocagne Identities. Finally, we present part of our studies involving the terms of r_{-n} and some partial sums of terms. With this work, we hope to encourage further studies on this class of numbers, providing new approaches to the repunit sequence and its various forms of representation.

ACKNOWLEDGEMENTS:

This work was partially supported by the PROPESQ-UFT.

7. REFERENCES

BEILER, A. H. *Recreations in the theory of numbers: The queen of mathematics entertains*. [S.l.]: Courier Corporation, 1964.

CARVALHO, P. C. P.; MORGADO, A. C. *Matemática discreta*. [S.l.]: SBM, Coleção PROFMAT. Rio de Janeiro, 2015.

CERDA, G. Matrix methods in horadam sequences. *Boletín de Matemáticas*, v. 19, n. 2, p. 97–106, 2012.

COSTA, E. A.; SANTOS, D. C. Algumas propriedades sobre os números monodígitos e repunidades. *Revista de Matemática da UFOP*, v. 2, p. 47–58, 2022.

COSTA, E. A.; SANTOS, D. C. dos. Algumas propriedades da sequência de pell. *CQD-Revista Eletrônica Paulista de Matemática*, v. 22, n. 3, p. 25–36, 2022.

DASDEMIR, A. Mersene, jacobstahl, and jacobstahl-lucas numbers with negative subscripts. *Acta Mathematica Universitatis Comenianae*, v. 88, n. 1, p. 142–156, 2019.

FALCON, S. On the generating matrices of the ê-fibonacci numbers. *Proyecciones (Antofagasta)*, SciELO Chile, v. 32, n. 4, p. 347–357, 2013.

HALICI, S.; AKYÜZ, Z. Fibonacci and lucas sequences at negative indices. *Konuralp Journal of Mathematics*, Mehmet Zeki SARIKAYA, v. 4, n. 1, p. 172–178, 2016.

HORADAM, A. F. Oresme numbers. *Fibonacci Quartely*, v. 20, n. 1, p. 73–76, 1982.

KALMAN, D. Generalized fibonacci numbers by matrix methods. *Fibonacci Quart*, v. 20, n. 1, p. 73–76, 1982.

KILIC, E. The generalized order-k fibonacci–pell sequence by matrix methods. *Journal of Computational and Applied Mathematics*, Elsevier, v. 209, n. 2, p. 133–145, 2007.

KILIC, E.; OEMUER, N.; ULUTAŞ, Y. Matrix representation of the second order recurrence $\{ukn\}$. *Ars Combin*, v. 93, p. 181–190, 2009.

KILIÇ, E.; STANICA, P. A matrix approach for general higher order linear recurrences. *Bull. Malays. Math. Sci. Soc.(2)*, v. 34, n. 1, p. 51–67, 2011.

KILIC, E.; TASCI, D.; HAUKKANEN, P. On the generalized lucas sequences by hessenberg matrices. *Ars Combin*, v. 95, p. 383–395, 2010.

MANGUEIRA, M. C. dos S. et al. As generalizações das formas matriciais e dos quatérnios da sequência de mersenne. *Revista de Matemática*, v. 1, n. 1, p. 1–17, 2021.

RIBENBOIM, P. *The little book of bigger primes*. [S.l.]: Springer, 2004. v. 811.

ROSEN, K. H. *Discrete mathematics and its applications*. [S.l.: s.n.], 2007.

SANTOS, D. C.; COSTA, E. A. Um passeio pela sequência repunidade. *CQD-Revista Eletrônica Paulista de Matemática*, v. 23, n. 1, p. 241–254, 2023.

SANTOS, D. C.; COSTA, E. A. Números repunidades e a representação matricial. *Revista Sergipana de Matemática e Educação Matemática*, n. 1, p. 81–96, 2024.

SLOANE, N. J. A. et al. *On-line encyclopedia of integer sequences*. AT & T Labs, 2024. Disponível em: <http://oeis.org/A002275>.

SOYKAN, Y. A study on generalized mersenne numbers. *Journal of Progressive Research in Mathematics*, v. 18, n. 3, p. 90–112, 2021.

SPREAFICO, E. V.; CRAVEIRO, I. M.; RACHIDI, M. Recorrências lineares homogêneas de ordem 2 com coeficientes constantes via funções geradoras ordinárias. *Revista Sergipana de Matemática e Educação Matemática*, v. 7, n. 1, p. 86–106, 2022.

TARASOV, B. V. The concrete theory of numbers: initial numbers and wonderful properties of numbers repunit. *arXiv preprint arXiv:0704.0875*, 2007.

TAŞTAN, M.; YILMAZ, N.; ÖZKAN, E. A new family of gauss (k, t)-horadam numbers. *Asian-European Journal of Mathematics*, World Scientific, v. 15, n. 12, p. 2250225, 2022.