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Fluxo turbulento: Inovações na análise de viscosidade

Turbulent flow: Innovations in viscosity analysis
Flujo turbulento: Innovaciones en el análisis de viscosidad

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Resumo
A análise matemática delineada neste estudo estabelece uma estrutura fundamental para explorar a regularidade das equações de Navier-Stokes. Dentro desse escopo, esta pesquisa representa um avanço significativo no uso do modelo de Smagorinsky em conjunto com a Simulação de Grandes Escalas (LES), culminando na formulação de uma nova aplicação baseada no Teorema enraizado em espaços de Banach e Sobolev. Embora a construção explícita de um modelo de viscosidade anisotrópico esteja além do escopo atual, esta aplicação lança as bases para seu desenvolvimento. Ao empregar análises matemáticas sofisticadas, este trabalho facilita uma compreensão mais abrangente em relação às complexidades que cercam os desafios de regularidade inerentes às equações de Navier-Stokes.

Palavras-chave: *Análise de Viscosidade. Regularidade. Equações de Navier-Stokes. Modelo de Smagorinsky.*

Abstract
The mathematical analysis outlined in this study establishes a fundamental framework for exploring the regularity of the Navier-Stokes equations. Within this scope, this research represents a significant advance in the use of the Smagorinsky model in conjunction with Large Eddy Simulation (LES), culminating in the formulation of a new application based on the Theorem rooted in Banach and Sobolev Spaces. Although the explicit construction of an anisotropic viscosity model is beyond the current scope, this application lays foundations for its development. By employing sophisticated mathematical analyses, this work facilitates a more comprehensive understanding regarding the complexities surrounding the regularity challenges inherent in the Navier-Stokes equations.

Keywords: *Viscosity Analysis. Regularity. Navier-Stokes Equations. Smagorinsky Model.*

Resumen
El análisis matemático descrito en este estudio establece un marco fundamental para explorar la regularidad de las ecuaciones de Navier-Stokes. Dentro de este ámbito, esta investigación representa un avance significativo en el uso del modelo de Smagorinsky en conjunto con la Simulación de Grandes Escalas (LES), culminando en la formulación de una nueva aplicación basada en el Teorema enraizado en los espacios de Banach y Sobolev. Si bien la construcción explícita de un modelo de viscosidad anisotrópica está más allá del alcance actual, esta aplicación sienta las bases para su desarrollo. Al emplear análisis matemáticos sofisticados, este trabajo facilita una comprensión más completa con respecto a las complejidades que rodean los desafíos de regularidad inherentes a las ecuaciones de Navier-Stokes.

Palabras-Clave: *Análisis de Viscosidad. Regularidad. Ecuaciones de Navier-Stokes. Modelo de Smagorinsky.*

1. INTRODUCTION

1.1. Understanding Turbulent Flow

Turbulent patterns manifest in both natural phenomena and human activities, such as river currents or plumes rising from chimneys. Analyzing the dynamics of turbulent motion is significant in fields like aeronautics, meteorology, and engineering. The Reynolds number, defined as

$$Re = \frac{UL}{\nu} = \frac{\rho UL}{\mu} \tag{1}$$

(where U is the characteristic velocity, L is the characteristic length, ν is the kinematic viscosity, ρ is the density, and μ is the dynamic viscosity), serves as a measure of the turbulence in a flow. Reynolds' experiment with pipe flow demonstrated that fluid motion with a Reynolds number exceeding 4×10^3 exhibits turbulence (for more details, see (SAGAUT, 2005)–(BREUER, 1998)).

1.2. Navier-Stokes Equations

In this study, the Navier-Stokes equations (NSE) play a pivotal role by offering a comprehensive depiction of fluid motion. Specifically, for incompressible and homogeneous fluids, these equations are expressed as follows:

$$\partial_t u_j + u_i \partial_i u_j = 2\nu \partial_i S_{ij} - \frac{1}{\rho}\partial_j P + f_j \quad \text{in} \quad \Omega \times (0,T], \quad j = 1,2,3 \tag{2}$$

$$\partial_t u_j = 0 \quad \text{in} \quad \Omega \times [0,T], \tag{3}$$

where $\mathbf{u}(x,t) = (u_1, u_2, u_3)(x,t)$ represents the velocity field dependent on position in space and time, ν denotes the kinematic viscosity,

$$S_{ij} = S_{ij}(\mathbf{u}) := \frac{1}{2}(\partial_i u_j + \partial_j u_i) , \tag{4}$$

is the rate of strain tensor, which signifies friction between particles (see (JOHN, 2014)), and $f = (f_1, f_2, f_3)$ represents forces per unit mass acting on the fluid. ρ stands for the fluid density, P denotes pressure, $(0,T]$ and $[0,T]$ denote time intervals, and $\Omega \subseteq \mathbb{R}^3$ denotes the domain. Equation (2) is based on the conservation of momentum, while Equation (3) is based on the conservation of mass.

1.3. Turbulent fluid flow

While the foundations supporting the derivation of the NSE are robust, it's essential to acknowledge that we are dealing with a model. One challenge arises from the inherent interdependence of velocity and pressure, while another stems from the nonlinearity of the convective term $u_i \partial_i u_j$ in Eq. (2).

Although this study does not primarily focus on conducting or demonstrating computational simulations, addressing the numerical solution of the NSE for turbulent flows remains intricate due to the vast amount of information encapsulated within the velocity field. These equations can be tackled through direct numerical simulations (DNS); however, the computational costs escalate rapidly, following a polynomial pattern relative to the Reynolds number. For instance, a DNS of a turbulent flow at $Re = 10^6$ would necessitate $Re^3 = 10^{18}$ uniformly distributed grid points in space-time, according to (JOHN, 2014).

As a result, computations involving extremely high Reynolds numbers remain impractical in the foreseeable future, despite advancements in Moore's Law. An alternative to direct numerical simulation (DNS) of non-averaged quantities involves a shift towards mean values, employing a statistical methodology. This shift is exemplified by large-eddy simulation (LES), which can be effectively implemented through the utilization of the Smagorinsky model. Unlike DNS, LES offers a more cost-effective approach, alleviating the constraints of DNS by explicitly calculating the dynamics of larger-scale motions while approximating the impact of smaller scales using simplified models. For further details, refer to (JOHN, 2014) and (CANT, 2001).

2. FUNDAMENTALS OF LES

2.1. Introduction to Large-Eddy Simlation

In the domain of large-eddy simulations (LES), significant macroscopic motions are directly represented, while small-scale motions undergo modeling. Pope (CANT, 2001) outlines four key conceptual steps:

a. The velocity $\mathbf{u}$ is split between a filtered component $\overline{\mathbf{u}}$ and a residual (subgrid-scale) component $\mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$. Ths former represents the motion of large eddies.

b. To ascertain the progression of the filtered velocity field, one derives the filtered Navier-Stokes equations from the original Navier-Stokes equations (NSE). These filtered equations mirror the structure of the unfiltered Navier-Stokes equations, with the inclusion of a residual stress tensor that emerges from the unresolved motions.

c. Modeling of the residual stress tensor becomes necessary to attain equation closure.

d. Subsequently, the filtered equations are numerically solved to determine the filtered velocity.

The filtering operation is characterized as

$$\overline{\mathbf{u}}\,(x,t) := \int G_{\Delta}\,(\mathbf{r},\mathbf{x})\,\mathbf{u}\,(\mathbf{x}-\mathbf{r},t)\,d\mathbf{r} \tag{5}$$

involving integration across the flow domain and the filter function G_{Δ} (frequently contingent on the filter width) that adheres to the normalization condition

$$\int G_{\Delta}\,(\mathbf{r},\mathbf{x})\,d\mathbf{r} = 1\,, \tag{6}$$

according to (CANT, 2001). Unless explicitly stated otherwise, an overline atop a variable indicates its filtered value.

A filter is called uniform if G_Δ does not depend on $\mathbf{x}$, and isotropic if G_Δ depends on $\mathbf{r}$ only through $r = [\mathbf{r}]$. Evidently, the filtering process maintains constants and adheres to linearity. Moreover, filtering demonstrates commutativity with both temporal differentiation and the computation of means, (CANT, 2001). Nonetheless, only specific filters exhibit commutativity when subjected to differentiation in relation to x_j, (see more at (JOHN, 2014)).

An often encountered isotropic filter comes in the form of a Gaussian

$$G_\Delta(\mathbf{r}) = \left(\frac{6}{\pi\Delta^2}\right)^{\frac{1}{2}} \exp\left(-\frac{6\,[\mathbf{r}]^2}{\Delta^2}\right), \tag{7}$$

according to (HOFFMAN; JOHNSON, 2006) and (CANT, 2001).

There are numerous filter functions with varying properties. We solely examine filters that commute with differentiation. Filtering the NSE Eq. (2) and Eq. (3), yields:

$$\partial_t \overline{u}_j + \overline{u}_i \partial_i \overline{u}_j = \partial_i\left(2\nu \overline{S}_{ij} - \tau^{\mathbf{r}}_{ij}\right) - \partial_j \overline{p} + \overline{f}_j \quad \text{in} \quad \Omega \times (0,T]\,, j = 1, 2.3\,, \tag{8}$$

$$\partial_t \overline{u}_j = 0 \text{ in } \Omega \times [0,T]\ . \tag{9}$$

While deriving the filtered continuity equation is straightforward, obtaining the filtered momentum equation necessitates some effort. The anisotropic residual-stress tensor $\tau^{\mathbf{r}}_{ij}$, is obtained by calculating the derivation of the filtered equation for momentum, performed by adapting what was done in the work of (CANT, 2001), we get, given that differentiation and filtering commute, and linearity is applicable in

$$\partial_t \overline{u}_j + \overline{u_i \partial_i u_j} = 2\nu \partial_i \overline{S}_{ij} - \frac{1}{\rho}\partial_j \overline{P} + \overline{f}_j \ \text{ in } \ \Omega \times (0,T]\,, j = 1, 2.3. \tag{10}$$

We establish the residual stress tensor, the anisotropic residual-stress tensor, and the adjusted filtered pressure:

$$\tau^R_{ij} := \overline{u_i u_j} - \overline{u}_i \overline{u}_j\ , \tag{11}$$

$$\tau^r_{ij} := \tau^R_{ij} - \frac{1}{3}\tau^R_{kk}\delta_{ij}\ , \tag{12}$$

$$\overline{p} := \frac{1}{\rho}\overline{P} + \frac{1}{3}\tau^R_{kk}\ . \tag{13}$$

Using the continuity Eqs. (3) and (9), we get:

$$\partial_i(u_i u_j) = (\partial_i u_i)\, u_j + u_i \partial_i u_j = u_i \partial_i u_j\ , \tag{14}$$

$$\partial_i(\overline{u}_i \overline{u}_j) = (\partial_i \overline{u}_i)\, \overline{u}_j + \overline{u}_i \partial_i \overline{u}_j = \overline{u}_i \partial_i \overline{u}_j\ . \tag{15}$$

Employing the preceding two equations along with the definition of the residual-stress tensor τ^R_{ij}, we obtain:

$$\begin{aligned}\partial_t\overline{u}_j \quad + \quad \overline{u_i\partial_i u_j} &= \partial_t\overline{u}_j \quad + \quad \overline{\partial_i\left(u_i u_j\right)} \\ &= \partial_t\overline{u}_j \quad + \quad \partial_i\left(\overline{u_i u_j}\right) \\ &= \partial_t\overline{u}_j \quad + \quad \partial_i\left(\overline{u}_i\overline{u}_j\right) + \partial_i\tau_{ij}^R \\ &= \partial_t\overline{u}_j \quad + \quad \overline{u}_i\partial_i\overline{u}_j + \partial_i\tau_{ij}^R\,.\end{aligned} \tag{16}$$

Now, with the incorporation of all three aforementioned definitions, we obtain:

$$\begin{aligned}-\partial_t\tau_{ij}^R - \frac{1}{\rho}\partial_j\overline{P} &= -\partial_t\tau_{ij}^r \quad - \quad \partial_i\frac{1}{3}\tau_{kk}^R\delta_{ij} - \frac{1}{\rho}\partial_j\overline{P} \\ &= -\partial_t\tau_{ij}^r \quad - \quad \partial_j\frac{1}{3}\tau_{kk}^R - \frac{1}{\rho}\partial_j\overline{P} \\ &= -\partial_t\tau_{ij}^r \quad - \quad \partial_j\left(\frac{1}{\rho}\overline{P} + \frac{1}{3}\tau_{kk}^R\right) \\ &= -\partial_t\tau_{ij}^r \quad - \quad \partial_j\overline{p}\,.\end{aligned} \tag{17}$$

Henceforth, we find ourselves at the juncture where the distilled equation of momentum unveils its form

$$\partial_t\overline{u}_j + \overline{u}_i\partial_i\overline{u}_j = \partial_i\left(2\nu\overline{S}_{ij} - \tau_{ij}^{\mathbf{r}}\right) - \partial_j\overline{p} + \overline{f}_j,\;\; j = 1,2,3. \tag{18}$$

3. SMAGORINKY SUB-GRID MODEL

To conclude the equations and consequently determine the filtered velocity field $\overline{\mathbf{u}}\left(\mathbf{x},t\right)$ along with the adjusted filtered pressure $\overline{p}\left(\mathbf{x},t\right)$, it is imperative to formulate the anisotropic residual stress tensor $\tau_{ij}^r\left(\mathbf{x},t\right)$. Among the available models, the Smagorinsky model stands out due to its simplicity and its demonstrated capability to yield satisfactory performance (more details at (CANT, 2001)).

In the Smagorinsky model, the anisotropic residual-stress tensor $\tau_{ij}^r\left(\mathbf{x},t\right)$ correlates with the filtered strain rate

$$\overline{S}_{ij} = \overline{S}_{ij}\left(\mathbf{u}\right) := S_{ij}\left(\overline{\mathbf{u}}\right) := \frac{1}{2}\left(\partial_j\overline{u}_i + \partial_i\overline{u}_j\right)\,, \tag{19}$$

as

$$\tau_{ij}^r\left(\mathbf{x},t\right) = -2\nu_r\overline{S}_{ij}\,. \tag{20}$$

This constitutes the mathematical embodiment of the Boussinesq hypothesis, which postulates that turbulent fluctuations exhibit dissipative behavior on average. The mathematical arrangement bears resemblance to that of molecular diffusion, (see more at (SAGAUT, 2005)). Substituting Eq. (20) into Eq. (8), the filtered momentum equation can be written as

$$\partial_t\overline{u}_j + \overline{u}_i\partial_i\overline{u}_j = 2\partial_i\left(\left(\nu+\nu_r\right)\overline{S}_{ij}\right) - \partial_j\overline{p} + \overline{f}_j,\;\; j = 1,2,3. \tag{21}$$

The residual subgrid-scale eddy-viscosity ν_r acts as an artificial viscosity (SAGAUT, 2005) and represents the eddy-viscosity of the residual motions. It is modeled as

$$\nu_r = \ell_S^2 \sqrt{\left(2\overline{S}_{lk}\overline{S}_{lk}\right)} = \left(C_S\Delta\right)^2 \sqrt{\left(2\overline{S}_{lk}\overline{S}_{lk}\right)}\,. \tag{22}$$

In this context, we encounter the Smagorinsky length scale $\ell_S = C_S\Delta$, the Smagorinsky coefficient C_S an the filter width Δ. Lastly, we can express the filtered momentum equation as follows

$$\partial_t\overline{u}_j + \overline{u}_i\partial_i\overline{u}_j = 2\partial_i\left(\left(\nu + \ell_S^2\sqrt{\left(2\overline{S}_{lk}\overline{S}_{lk}\right)}\right)\overline{S}_{ij}\right) - \partial_j\overline{p} + \overline{f}_j,\;\; j = 1,2,3. \tag{23}$$

The model for the eddy-viscosity, Eq. (22), is called Smagorinsky model.

The Smagorinsky model comes with certain limitations. They are summarized as follows in:

a. The Smagorinsky model constant C_S is an a priori input. The single contant is incapable to represent correctly various turbulent flows;

b. The eddy-viscosity does not vanish for a laminar flow;

c. The backscatter of energy is prevented completely since

$$\left(C_S\Delta\right)^2 \sqrt{\left(2\overline{S}_{lk}\overline{S}_{lk}\right)} \geq 0\,.$$

d. The Smagorinsky model typically introduces excessive diffusion into the flow.

4. DYNAMICS IN THE SMAGORINSKY MODEL

4.1. Derivation of the Smagorinsky model

Based on (FERZIGER; PERIĆ; STREET, 2019), the Smagorinsky model's derivation can occur through various approaches, such as heuristic techniques. For instance, one method involves equating the production and dissipation of subgrid-scale turbulent kinetic energy. Alternatively, the model can be derived using turbulence theories. The formulation (derivation) presented here has been adapted from (SAGAUT, 2005). Both heuristic approaches and turbulence theories are given consideration. Kolmogorov (KOLMOGOROV, 1991), (cited in (FERZIGER; PERIĆ; STREET, 2019)) attained the generalized expression for the energy spectrum function

$$E\left(k\right) = K\left\langle\varepsilon\right\rangle^{\frac{2}{3}} k^{-\frac{5}{3}}\,, \quad K \approx 1.4\,, \tag{24}$$

where

$$\varepsilon\left(t\right) := \left|\Omega\right|^{-1} \int_\Omega \nu\left|\nabla\mathbf{u}\right|^2\left(\mathbf{x}, t\right) d\mathbf{x}\,, \tag{25}$$

is the energy dissipation rate, K a constant and angular brackets indicate a statistical mean. In other words, this signifies an energy cascade from the larger scales to the smaller scales. This has been famously summarized in a poem by mathematician and meteorologist L. F. Richardson, as quoted in (FERZIGER; PERIĆ; STREET, 2019): *"Big whirls have little whirls what feed on their velocity, little whirls have smaller whirls, and so on to viscosity."*

Dimensional investigation reveals that

$$\partial_i \tau_{ij}^r \left[\frac{m}{s^2}\right] \Longleftrightarrow \tau_{ij}^r = -2\nu_r \overline{S}_i \left[\frac{m^2}{s^2}\right] \Longleftrightarrow \nu_r \left[\frac{m^2}{s}\right] .$$

Therefore, it is assumed that the residual subgrid-scale eddy viscosity ν_r is proportional to $\varepsilon^{\sim\frac{1}{3}}\Delta^{\frac{4}{3}}$ the kinetic energy transfer rate (see more in (SAGAUT, 2005)). Using Eq. (24) and the so-called two-fluid model or eddy-damped quasinormal Markovian model, we get

$$\langle \nu_r \rangle = \frac{A}{\pi^{\frac{4}{3}} K} \langle \widetilde{\varepsilon} \rangle^{\frac{1}{3}} \Delta^{\frac{4}{3}} , \tag{26}$$

where A is a constant, wich is 0.438 according to the two-fluid model and 0.441 according to the eddy-damped quasinormal Markovian model, both cited in (SAGAUT, 2005). Furthermore, in the isotropic homogeneous case,

$$\left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle = \int_0^{\frac{\pi}{\Delta}} 2k^2 E(k)\, dk \tag{27}$$

is true, according (SAGAUT, 2005). Substituting Eq. (24) into Eq. (27), yields

$$\begin{aligned} \left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle &= \int_0^{\frac{\pi}{\Delta}} 2k^2 K (\varepsilon)^{\frac{2}{3}} k^{-\frac{5}{3}} dk \\ &= 2K \langle \varepsilon \rangle^{\frac{2}{3}} \int_0^{\frac{\pi}{\Delta}} k^{\frac{1}{3}} dk \\ &= \frac{3}{2} K \langle \varepsilon \rangle^{\frac{2}{3}} \pi^{\frac{4}{3}} \Delta^{-\frac{4}{3}} . \end{aligned} \tag{28}$$

This is equivalent to

$$\begin{aligned} \left(\frac{3K}{2}\right)^{\frac{3}{2}} \langle \varepsilon \rangle \pi^2 \Delta^{-2} &= \left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle^{\frac{3}{2}} \\ \Longleftrightarrow \qquad \langle \varepsilon \rangle &= \pi^{-2} \left(\frac{3K}{2}\right)^{-\frac{3}{2}} \Delta^2 \left\langle 2\overline{S}_{lk}\overline{S}_{lk} \right\rangle^{\frac{3}{2}} . \end{aligned} \tag{29}$$

This formulation means to say that, the local equilibrium hypothesis states that the flow is in a constant spectral equilibrium. As a result, energy does not accumulate at any frequency, and the shape of the energy spectrum remains unchanged over time. This implies that the production, dissipation, and energy flux through the cutoff are all equal

$$\langle \varepsilon_I \rangle = \langle \tilde{\varepsilon} \rangle = \langle \varepsilon \rangle . \tag{30}$$

Using the last equation, we can insert Eq. (29) into Eq. (26) and get

$$\begin{aligned}
\langle \nu_r \rangle &= \frac{A}{\pi^{\frac{4}{3}} K} \langle \widetilde{\varepsilon} \rangle^{\frac{1}{3}} \Delta^{\frac{4}{3}} \\
&= \frac{A}{\pi^{\frac{4}{3}} K} \langle \varepsilon \rangle^{\frac{1}{3}} \Delta^{\frac{4}{3}} \\
&= \frac{A}{K} \pi^{-\frac{4}{3}} \left(\pi^{-2} \left(\frac{3K}{2} \right)^{-\frac{3}{2}} \Delta^2 \left\langle 2 \overline{S}_{lk} \overline{S}_{lk} \right\rangle^{\frac{3}{2}} \right) \Delta^{\frac{4}{3}} \\
&= \frac{A}{K} \pi^{-2} \left(\frac{3K}{2} \right)^{-\frac{3}{2}} \Delta^2 \left\langle 2 \overline{S}_{lk} \overline{S}_{lk} \right\rangle^{\frac{1}{2}} .
\end{aligned}$$

Defining the Smagorinsky coeffcient as

$$C_S := A^{1/2} \left(\pi^{-1} K^{-\frac{1}{2}} \right) \left(\frac{3K}{2} \right)^{-\frac{1}{4}} \approx 0.148 \,, \tag{31}$$

we can write,

$$\langle \nu_r \rangle = (C_S \Delta)^2 \left\langle 2 \overline{S}_{lk} \overline{S}_{lk} \right\rangle^{\frac{1}{2}} . \tag{32}$$

The Smagorinsky model is then expressed as

$$\nu_r (\mathbf{x}, t) = (C_S \Delta)^2 \left(2 \overline{S}_{ij} (\mathbf{x}, t) \, \overline{S}_{ij} (\mathbf{x}, t) \right)^{\frac{1}{2}} . \tag{33}$$

Sagaut acknowledges that this proposition lacks specific justification, other than its observed average validity as demonstrated in Eq. (32) (cited in (SAGAUT, 2005)). The model's validation stems from its performance. Pope at (CANT, 2001), at the very least, deems it satisfactory, although he highlights subpar outcomes in specific scenarios. It is important to note that the Smagorinsky coefficient C_S was evaluating in Eq. (31), but is adjusted to improve results. Through different analysis, the values 0.17 ((CANT, 2001)), 0.18 ((KOLMOGOROV, 1991)) and 0.15 were obtained as well. Opting for C_S to vary with both space and time, rather than remaining a constant, could potentially yield even more favorable results. This will be addressed in the following section.

4.2. Dynamic Smagorinsky model

To consider the Smagorinsky coefficient as a function of space and time (based on what was done in (BREUER, 1998)), we propose an idea that is presented here. Using a so-called test filter $\hat{\Delta} > \Delta$, the filtered NSE (Eq. (2) and (3)) are filtered again:

$$\partial_t \hat{\bar{u}}_j + \partial_i \left(\widehat{\bar{u}_i \bar{u}_j} \right) = \partial_i \left(2\nu \hat{\bar{S}}_{ij} - \hat{\tau}^{\mathbf{r}}_{ij} \right) - \partial_j \hat{\bar{p}} + \hat{\bar{f}}_j, \quad j = 1, 2.3, \quad \text{in } \Omega \times (0, T] \, ,$$

$$\partial_t \hat{\bar{u}}_j = 0, \qquad \text{in } \Omega \times [0, T] \, ,$$

with hats indicating the second filtering. Similar to the residual-stress tensor τ^R_{ij} is defined as

$$\tau^R_{ij} := \overline{u_i u_j} - \bar{u}_i \bar{u}_j \, ,$$

the subtest-scale stress-tensor ${}_s\mathbb{T}_{ij}$ is defined as

$$ {}_s\mathbb{T}_{ij} = \widehat{\overline{u_i u_j}} - \hat{\bar{u}}_i \hat{\bar{u}}_j \, , \tag{34}$$

such that

$$\mathbb{G}_{ij} = {}_s\mathbb{T}_{ij} - \widehat{\tau^R_{ij}} = \widehat{\bar{u}_i \bar{u}_j} - \hat{\bar{u}}_i \hat{\bar{u}}_j \tag{35}$$

$\mathbb{G}_{ij}$ is called the Germano identidy (see more at (LILLY, 1992)). We denote the Smagorinsky parameter with $\widetilde{C}_S$ (instead of C_S). It is formulated without an exponent in the assumption, unlike the Smagorinsky coefficient in the Smagorinsky model (Eq. (22)).

The approach taken is (c. f. Eqs. (20) and (22)):

$$\tau^r_{ij} := \tau^R_{ij} - \frac{1}{3}\tau^R_{kk}\delta_{ij} = -2\widetilde{C}_S\left(\mathbf{x}, t\right)\Delta^2 \sqrt{\left(2\bar{S}_{lk}\bar{S}_{lk}\right)}\bar{S}_{ij} \, ,$$

$$ {}_s\mathbb{T}_{ij} - \frac{1}{3}\, {}_s\mathbb{T}_{kk}\delta_{ij} = -2\widetilde{C}_S\left(\mathbf{x}, t\right)\Delta^2 \sqrt{\left(2\bar{S}_{lk}\bar{S}_{lk}\right)}\bar{S}_{ij} \, .$$

So, to get an equation for $\widetilde{C}_S$, it is necessary to approximate

$$\tau^r_{ij} \approx -2\widetilde{C}_S\left(\mathbf{x}, t\right)\Delta^2 \left[\widehat{\sqrt{\left(2\bar{S}_{lk}\bar{S}_{lk}\right)}\bar{S}_{ij}} \right] . \tag{36}$$

Equality is achieved when the Smagorinsky parameter is not dependent on $\mathbf{x}$, c. f. (KOLMOGOROV, 1991). For $i, j \in \{1, 2, 3\}$, the system

$$\begin{aligned} \mathbb{G}_{ij} - \frac{1}{3}\mathbb{G}_{kk}\delta_{ij} &= {}_s\mathbb{T}_{ij} - \frac{1}{3}\, {}_s\mathbb{T}_{kk}\delta_{ij} - \widehat{\tau^r_{ij}} \\ &\approx -2\widetilde{C}_S \left(\hat{\Delta}^2 \sqrt{\left(2\hat{\bar{S}}_{lk}\hat{\bar{S}}_{lk}\right)}\hat{\bar{S}}_{lk} - \Delta^2 \widehat{\sqrt{\left(2\bar{S}_{lk}\bar{S}_{lk}\right)}\bar{S}_{ij}} \right) \\ &= -2\widetilde{C}_S \mathbb{M}_{ij} \, , \end{aligned}$$

with

$$\mathbb{M}_{ij} := \hat{\Delta}^2 \sqrt{\left(2\hat{\bar{S}}_{lk}\hat{\bar{S}}_{lk}\right)}\hat{\bar{S}}_{lk} - \Delta^2 \widehat{\sqrt{\left(2\bar{S}_{lk}\bar{S}_{lk}\right)}\bar{S}_{ij}} \, ,$$

is an overdetermined system which $\widetilde{C}_S$ cannot satisfy exactly.

D. K. Lilly (LILLY, 1992), therefore, propo-se a least-square method, minimizing the square of the error

$$Q = \left(\mathbb{G}_{ij} - \frac{1}{3}\mathbb{G}_{kk}\delta_{ij} + 2\widetilde{C}_S\mathbb{M}_{ij}\right)^2 \, ,$$

meaning the sum over all i, j. Since

$$\begin{aligned} \frac{\partial Q}{\partial \widetilde{C}_S} &= 2\left(\mathbb{G}_{ij} - \frac{1}{3}\mathbb{G}_{kk}\delta_{ij} + 2\widetilde{C}_S\mathbb{M}_{ij}\right)2\mathbb{M}_{ij} \\ &= 4\mathbb{G}_{ij}\mathbb{M}_{ij} - \frac{4}{3}\mathbb{G}_{kk}\delta_{ij}\mathbb{M}_{ij} + 8\widetilde{C}_S\mathbb{M}_{ij}\mathbb{M}_{ij} \\ &= 4\mathbb{G}_{ij}\mathbb{M}_{ij} - \frac{4}{3}\mathbb{G}_{kk}\delta_{ij}\mathbb{M}_{ll} + 8\widetilde{C}_S\mathbb{M}_{ij}\mathbb{M}_{ij} \\ &= 4\mathbb{G}_{ij}\mathbb{M}_{ij} + 8\widetilde{C}_S\mathbb{M}_{ij}\mathbb{M}_{ij} \, , \end{aligned}$$

with

$$\frac{\partial^2 Q}{\partial \widetilde{C}_S^2} = 8\mathbb{M}_{ij}\mathbb{M}_{ij} > 0 \, ,$$

note that $\mathbb{M}_{ll} = 0$ because $\hat{\bar{S}}_{ll} = 0$ and $\bar{S}_{ll} = 0$, see Eq. (9). The Smagorinsky parameter minimizes the error when we establish

$$\widetilde{C}_S(\mathbf{x}, t) = -\frac{\mathbb{G}_{ij}\mathbb{M}_{ij}}{2\mathbb{M}_{ij}\mathbb{M}_{ij}}(\mathbf{x}, t) \, . \tag{37}$$

5. MATHEMATICAL ANALYSIS OF THE SMAGORINSKY MODEL

Based on the work of (BREZIS; BRÉZIS, 2011) and (SHOWALTER, 2013), a mathematical analysis for Smagorinsky's model was essential, allowing the problem to be defined clearly and precisely.

5.1. Vector spaces

The Lebesgue space $L^p(\Omega)$, $p \in [1, \infty]$, is the Banach space of measurable functions $\mathbf{v}$ on Ω which satisfy

$$\|\mathbf{v}\|_{L^{m,p}(\Omega)} := \left(\int_{\Omega} |\mathbf{v}(\mathbf{x})|\, d\mathbf{x}\right)^{\frac{1}{p}} < \infty\,, \quad \text{if} \quad p \in [1,\infty)\,, \tag{38}$$

$$\|\mathbf{v}\|_{L^{m,p}(\Omega)} := \text{ess sup}\, |\mathbf{v}(\mathbf{x})| < \infty\,, \qquad \text{if} \qquad p = \infty\,,$$

For $p = 2$, the Labesgue space is also a Hilbert space with the scalar product

$$(\mathbf{x}, \mathbf{v}) = \int_{\Omega} \mathbf{v}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x}\,.$$

in the case of one-dimensional functions, the dot signifies straightforward multiplication; however, when dealing with vectors or matrices, it denotes the dot product for vectors or the Frobenius inner product for matrices. For two matrices $A = (a_{ij})_{1\leq i,j\leq 3}$ and $B = (b_{ij})_{1\leq i,j\leq 3}$, the Frobenius inner product is

$$A : B := a_{ij} b_{ij}\,.$$

We write $L^p(a, b; V)$ for the Lebesgue space of functions from the interval (a, b) to the Banach space V. The identical notation is employed for the corresponding Sobolev spaces.

The Sobolev space $W^{m,p}$ is the Banach space of functions for wich

$$\|\mathbf{v}\|_{W^{m,p}(\Omega)} := \left(\sum_{0\leq|\alpha|\leq m} \|D^{\alpha}\mathbf{v}\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} < \infty\,, \quad \text{if} \quad p \in [1,\infty)\,,$$

$$\|\mathbf{v}\|_{W^{m,p}(\Omega)} := \max_{0\leq|\alpha|\leq m} \|D^{\alpha}\mathbf{v}\|^p_{L^p(\Omega)} < \infty\,, \qquad \text{if} \qquad p = \infty\,,$$

remains valid, i.e., it can be defined as

$$W^{m.p}(\Omega) = \{\mathbf{v} \in L^p(\Omega) : D^{\alpha}\mathbf{v} \in L^p(\Omega)\ \forall\, |\alpha| \leq m\}\,. \tag{39}$$

Let

$$W^{1.3}_{0,\text{div}}(\Omega) = \left\{\mathbf{v} \in W^{1,3}(\Omega) : \mathbf{v}|_{\Gamma} = 0, \nabla \cdot \mathbf{v} = 0 \,\text{in}\, \Omega\right\}\,, \tag{40}$$

be the divergence-free Sobolev space where functions vanish on the boundary $\Gamma = \partial\Omega$,

$$H^1\left(0, T; L^2(\Omega)\right) := W^{1,2}\left(0, T; L^2(\Omega)\right)\,,$$

a Sobolev space that is also a Hilbert space and

$$V := H^1\left(0, T; L^2(\Omega)\right) \cap L^3\left(0, T; W^{1.3}_{0,\text{div}}(\Omega)\right)\,, \tag{41}$$

a Banach space with the norm

$$\|\mathbf{v}\|_V = \|\nabla\mathbf{v}\|_{L^3(0,T;L^3(\Omega))} + \|\partial_t \mathbf{v}\|_{L^2(0,T;L^2(\Omega))}\,.$$

5.2. Weak formulation of the problem

Consider the NSE with the conditions

$$\begin{aligned}
\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = \nu\nabla\cdot\nabla\mathbf{u} - \frac{1}{\rho}\nabla P + f \qquad &\text{in} \quad \Omega\times(0,T]\,,\\
\nabla\cdot\mathbf{u} = 0 \qquad &\text{in} \quad \Omega\times[0,T]\,,\\
\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \qquad &\text{in} \qquad \Omega,\\
\mathbf{u} = 0 \qquad &\text{on} \quad \Gamma\times[0,T]\,,\\
\int_\Omega P\,d\mathbf{x} = 0 \qquad &\text{in} \qquad (0,T]\,.
\end{aligned} \tag{42}$$

with $\Gamma = \partial\Omega$. Note that

$$2\partial_i S_{ij} = \partial_i\left(\partial_i u_j + \partial u_i\right) = \partial_i\partial_i u_j + \partial_i\partial_j u_i = \partial_i\partial_i u_j + \partial_j\left(\partial_i u_i\right) = \partial_i\partial_i u_j = \nabla\cdot\nabla\mathbf{u}\,.$$

The first and second equations correspond to the momentum equation (Eq. (2)) and continuity equation (Eq. 3) from above. The initial flow field $\mathbf{u}_0(\mathbf{x})$ is also divergence-free, i.e., $\nabla\cdot\mathbf{u}_0 = 0$ in Ω. The fourth equation is the no-slip boundary condition. It relies on the supposition that the fluid does not permeate or slide along the wall. Without the last equation, the pressure P would only be determined up to a constant, (JOHN, 2014).

Filtering Eqs. (42) and using a similar condition for the modified filtered pressure, we get

$$\begin{aligned}
\partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}}\cdot\nabla)\,\overline{\mathbf{u}} = \nabla\cdot(\nu+\nu_r)\,\nabla\overline{\mathbf{u}} - \nabla\overline{p} + \overline{\mathbf{f}} \qquad &\text{in} \quad \Omega\times(0,T]\,,\\
\nabla\cdot\overline{\mathbf{u}} = 0 \qquad &\text{in} \quad \Omega\times[0,T]\,,\\
\overline{\mathbf{u}}(\mathbf{x},0) = \overline{\mathbf{u}}_0(\mathbf{x}) \qquad &\text{in} \qquad \Omega,\\
\overline{\mathbf{u}} = 0 \qquad &\text{on} \quad \Gamma\times[0,T]\,,\\
\int_\Omega \overline{p}\,d\mathbf{x} = 0 \qquad &\text{in} \qquad (0,T]\,.
\end{aligned} \tag{43}$$

By multiplying the first equation with $\mathbf{v}\in V$ and integrating over time and space, we achieve a weak formulation. Let $\overline{\mathbf{f}} \in L^2\left(0,T;L^2(\Omega)\right)$. Find $\overline{\mathbf{u}}\in V$ that satisfies $\overline{\mathbf{u}} = (0,\mathbf{x}) = \overline{\mathbf{u}}_0 \in$

$W^{1,3}_{0,\text{div}}(\Omega)$ and

$$\int_0^T (\partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}}, \mathbf{v}) + ((\nu + \nu_r)\nabla\overline{\mathbf{u}}, \nabla \mathbf{v})\, dt = \int_0^T (\overline{\mathbf{f}}, \mathbf{v})\, dt\,, \tag{44}$$

for all $\mathbf{v} \in V$, with $(\cdot,\cdot)$ denoting the $L^2\left(0,T;L^2(\Omega)\right)$ scalar product. Let $\mathbf{n}$ be the outward unit surface normal to $\Gamma = \partial\Omega$. Note that using integration by parts, we can derive:

$$\begin{aligned}(\nabla \cdot \mathbf{w}, \mathbf{v}) &= \int_\Omega (\nabla\cdot\mathbf{w})\cdot\mathbf{v}d\mathbf{x} = \int_\Gamma (\mathbf{w}\cdot\mathbf{n})\cdot\mathbf{v}ds - \int_\Omega \mathbf{w}\cdot(\nabla\mathbf{v})\,d\mathbf{x} \\ &= -\int_\Omega \mathbf{w}\cdot(\nabla\mathbf{v})\,d\mathbf{x} = -\mathbf{w}\cdot(\nabla\mathbf{v})\,,\end{aligned}$$

because $\mathbf{v} = 0$ on Γ. In this case, we used $\mathbf{w} = (\nu + \nu_r)\nabla\overline{\mathbf{u}}$. The pressure term vanishes because

$$(\nabla\overline{p}, \mathbf{v}) = \int_\Omega \nabla\overline{p}\cdot\mathbf{v}d\mathbf{x} = \int_\Gamma \overline{p}\,(\mathbf{v}\cdot\mathbf{n})\,ds - \int_\Omega \overline{p}\,(\nabla\cdot\mathbf{v})\,d\mathbf{x} = 0\,,$$

as $\nabla\cdot\mathbf{v} = 0$. Another similar variation of this following formulation. Find $(\mathbf{w}, q) : [0,T] \to \mathbf{X}\times Q$ satisfying $\mathbf{w}(\mathbf{x}, 0) = \overline{\mathbf{u}}_0(\mathbf{x})$ and

$$(\partial_t\mathbf{w}, \mathbf{v}) + a(\mathbf{u}, \mathbf{w}, \mathbf{v}) + b(\mathbf{u}, \mathbf{w}, \mathbf{v}) + (\lambda, \nabla\cdot\mathbf{w}) - (q, \nabla\cdot\mathbf{v}) = \left(\overline{\mathbf{f}}, \mathbf{v}\right)\,, \tag{45}$$

for all $(\mathbf{v}, \lambda) \in \mathbf{X}\times Q$ with

$$\begin{aligned} a(\mathbf{u}, \mathbf{w}, \mathbf{v}) &:= \alpha(\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v}) \\ &\quad + \left(\left(2Re^{-1} + \widetilde{C}_S\Delta^2\left(\sqrt{\overline{S}_{lk}\overline{S}_{lk}}\right)\right)\overline{S}(\mathbf{w}), \overline{S}(v)\right) \\ b(\mathbf{u}, \mathbf{w}, \mathbf{v}) &:= \frac{1}{2}(\mathbf{u}, \nabla\mathbf{w}, \mathbf{v}) - \frac{1}{2}(\mathbf{u}, \nabla\mathbf{v}, \mathbf{w})\,. \end{aligned}$$

with the Smagorinsky parameter $\widetilde{C}_S = \sqrt{2}C_S^2$, according to (KOLMOGOROV, 1991).

5.3. Behavior and Regularity

Let's start by introducing several standard notations and function spaces that will be employed in the following analysis. Specifically, we denote: $\mathcal{V} = \left\{\varphi \in \mathcal{D}(\Omega)^3, \nabla\cdot\varphi = 0\right\}$, H as the closure of $\mathcal{V}$ in $L^2(\Omega)^3$, and V as the closure of $\mathcal{V}$ in $W^{1,3}(\Omega)^3$, where $L^2(\Omega)^3$ is the space of functions that are square integrable over Ω with respect to the Lebesgue measure, and $W^{1,3}(\Omega)^3$ is the Sobolev space $W^{1,3}$ with vector-valued functions. The space H is a Hilbert space with respect to the inner product. We will use the notation V' for the dual space of V, $\|\cdot\|_V$ for the induced norm, and $\langle\cdot,\cdot\rangle$ for the duality product.

For spaces of functions which depend on both time and space variables, we will frequently use the twofollowing spaces: **(a)** C$([0,T]\,;X)$ the space of continuous functions $u : [0,T] \to X$, where X is a Banach space with the norm denoted by $|\cdot|_X$. (b) $L^p\left(0,T;X\right)$ the space of strongly measurable functions $u:]0,T[\rightarrow X$ with a finite norm

$$|u|^p_{L^p(0,T,X)} := \int_0^T |u\left(t\right)|^p_X \, dt < \infty.$$

In the case $p=\infty$ the norm is defined by

$$|u|^p_{L^\infty(0,T,X)} := \text{ess } \sup_{t\in]0,T[} |u\left(t\right)|_X \,.$$

Finally, we will denote by $|\cdot|_p$ the usual norm in $L^p\left(\Omega\right)$.

Lemma 5.1. *Let X be a Banach space and X_0, X_1 two reflexive, separable Banach spaces. If we assume that*

$$X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$$

the first embedding being compact, then we have the following embedding

$$\left\{v\in L^\alpha\left(0,T;X_0\right), \frac{dv}{dt}\in L^\beta\left(0,T;X_1\right)\right\}\hookrightarrow\hookrightarrow L^\alpha\left(0,T;X\right)\,,$$

where $1<\alpha,\beta<\infty$.

The proof this lemma can be found in (LIONS, 1969).

In this context, we consider the weak fomuation for the problem (42). Derived from multiplying the momentum equation by a test function and applying integration by parts, resulting in the issue that will be mentioned in the sequel as Problem: For $\mathbf{f}\in L^{\frac{3}{2}}\left(0,T;V'\right)$ and $\mathbf{u}_0\in H$ given, find $\mathbf{u}$ satisfying

$$\mathbf{u}\in C\left(\left[0,T\right];H\right)\cap L^3\left(0,T;V\right) \text{ with } \frac{d\mathbf{u}}{dt}\in L^{\frac{3}{2}}\left(0,T;V'\right)\,,$$

$$\left\langle\frac{d\mathbf{u}}{dt},\mathbf{v}\right\rangle+\sum_{i,j=1}^{3}\int_\Omega \mathbb{T}_{ij}\left(\mathbf{S}\left(\mathbf{u}\right)\right)S_{ij}\left(\mathbf{v}\right)d\mathbf{x}+\int_\Omega\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}\cdot\mathbf{v}d\mathbf{x}=\left\langle\mathbf{f},\mathbf{v}\right\rangle,\forall\mathbf{v}\in V \tag{46}$$

and the initial condition

$$\mathbf{u}\left(0\right)=\mathbf{u}_0\,.$$

The following application is based on the Theorem presented and demonstrated in the work of (SANTOS; SALES, 2023), pg. 5.

Application. *Let $\mathbf{u}_0\in H$ and $\mathbf{f}\in L^{\frac{3}{2}}\left(0,T;V'\right)$. Then for any $\mathbb{T}>0$ the problem has a unique weak solution on $[0,T]$. Moreover, if $\mathbf{u}_0\in V$ then the unique weak solution is in $L^\infty\left(0,T;W^{1,3}\left(\Omega\right)^3\right)$.*

To prove the existence of a weak solution we used a classical Galerkin method. We omit it, since it is straightforward from the proof done in (LIONS, 1969) based on the compactness method. A complete demonstrationcan be found in (JIROVEANU, 2002). We only present here the proof of uniqueness.

Let us suppose that there exist two weak solutions $\mathbf{u}$ and $\mathbf{v}$ to problem (S), with the same initial condition $\mathbf{u}_0\in H$ and let $\mathbf{w}=\mathbf{u}-\mathbf{v}$. After subtracting the weak formulation for $\mathbf{v}$ from the one for $\mathbf{u}$ and talking $\mathbf{w}$ as test function in the resulting equation, we get

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|_2^2+\sum_{i,j=1}^{3}\int_\Omega\left[\mathbb{T}_{ij}\left(\mathbf{S}\left(\mathbf{u}\right)\right)-\mathbb{T}_{ij}\left(\mathbf{S}\left(\mathbf{v}\right)\right)\right]S_{ij}\left(\mathbf{w}\right)d\mathbf{x}=-\int_\Omega\left(\mathbf{w}\cdot\nabla\right)\mathbf{u}\mathbf{w}d\mathbf{x}\,. \tag{47}$$

Moreover, from the definition of the tensor $\mathbb{T}$, we have

$$\sum_{i,j=1}^{3}\int_{\Omega}\left[\mathbb{T}_{ij}\left(\mathbf{S}\left(\mathbf{u}\right)\right)-\mathbb{T}_{ij}\left(\mathbf{S}\left(\mathbf{v}\right)\right)\right]d\mathbf{x}\geq c_{1}\sum_{i,j=1}^{3}\int_{\Omega}\left|S_{ij}\left(\mathbf{w}\right)\right|^{2}d\mathbf{x}\,, \tag{48}$$

with $c_1 > 0$. Now, using Korn's inequality

$$\left(\int_{\Omega}\left|\mathbf{S}\left(\mathbf{u}\right)\right|^{p}d\mathbf{x}\right)^{\frac{1}{p}}\geq C_{p}\left|\nabla\mathbf{u}\right|_{p}\,,$$

for $\mathbf{u}\in W_{0}^{1,p}$ with $C_p > 0$ $(1<p<\infty)$ and Hölder's inequality we obtain from Eq. (48):

$$\frac{1}{2}\frac{d}{dt}\left|\mathbf{w}\right|_{2}^{2}+c_{2}\left|\nabla\mathbf{w}\right|_{2}^{2}\leq\int_{\Omega}\left|\mathbf{w}\right|^{2}\left|\nabla\mathbf{u}\right|d\mathbf{x}\leq\left|\nabla\mathbf{u}\right|_{3}\left|\mathbf{w}\right|_{3}^{2}\,. \tag{49}$$

In three dimensions we have the embedding

$$H^{1}\left(\Omega\right)\subset L^{6}\left(\Omega\right)$$

from which we deduce

$$\left|\mathbf{w}\right|_{3}\leq\left|\mathbf{w}\right|_{2}^{\frac{1}{2}}\left|\mathbf{w}\right|_{6}^{\frac{1}{2}}\leq c_{3}\left|\mathbf{w}\right|_{2}^{\frac{1}{2}}\left|\nabla\mathbf{w}\right|_{2}^{\frac{1}{2}}\,.$$

Moreover, it follows from Eq. (49), via Young's inequality, that

$$\frac{d}{dt}\left|\mathbf{w}\right|_{2}^{2}+c_{4}\left|\nabla\mathbf{w}\right|_{2}^{2}\leq c_{5}\left|\nabla\mathbf{u}\right|_{3}^{2}\left|\mathbf{w}\right|_{2}^{2}\,. \tag{50}$$

Since the function $g\left(t\right)=\left|\nabla\mathbf{u}\right|_{3}^{2}$ is integrable on $]0,T[$ and $\mathbf{w}\left(0\right)=0$, using Gronwall's inequality we get

$$\left|\mathbf{w}\left(t\right)\right|_{2}^{2}=0\,,$$

on $[0,T]$ and thus uniqueness of the solution to problem.

The uniform in time regularity is related to the asymptotic behavior of the solution that we now consider. Let $\mathbf{u}_0\in H$ and suppose now that $\mathbf{f}\in L^{2}\left(\Omega\right)^{3}$ is time independent. According Theorem **??** the unique weak solution is continous

$$\mathbf{u}\in C\left(\left(0,T\right);H\right).$$

Consequently, we can define the family of operators $\left(S\left(t\right)\right)_{t\geq0}$ by

$$\begin{aligned} S\left(t\right):\quad & H\rightarrow H\\ & \mathbf{u}_{0}\mapsto S\left(t\right)_{\mathbf{u}_{0}=\mathbf{u}\left(t\right)} \end{aligned} \tag{51}$$

is the solution to problem. It is easy to show that this family form a continuous semigroup for which we have

Proposition 5.2. *The exists a ball*

$$B_{p}=\left\{\mathbf{u}\in V;\left|\nabla\mathbf{u}\right|_{3}\leq\rho\right\}$$

which absorbs all the balls in the space H.

Proof. Taking in Eq. (46) $\mathbf{u}$ as test function and using the property

$$\int_{\Omega}(\mathbf{u}\cdot\nabla)\,\mathbf{u}\cdot\mathbf{u}\,d\mathbf{x},\forall\mathbf{u}\in V$$

we obtain

$$\frac{d}{dt}\left|\mathbf{u}\right|_2^2+\sum_{i,j=1}^{3}\int_{\Omega}\mathbb{T}_{ij}\left(\mathbf{S}\left(\mathbf{u}\right)\right)S_{ij}\left(\mathbf{u}\right)d\mathbf{u}=\left\langle\mathbf{f},\mathbf{u}\right\rangle. \tag{52}$$

The tensor $\mathbb{T}_{ij}$ can be represented through a nonnegative potential $\theta:\mathbf{R}^9\to\mathbf{R}$ given by

$$\theta\left(\mathbf{S}\right)=\int_0^{\left|\mathbf{S}\right|^2}\left(\nu+\nu_1\sqrt{\mathbf{y}}\right)d\mathbf{y}. \tag{53}$$

Indeed, we have

$$\mathbb{T}_{ij}\left(\mathbf{S}\right)=\frac{\partial\theta\left(\mathbf{S}\right)}{\partial S_{ij}},\forall\,i,j=1,2,3.$$

Moreover,

$$\theta\left(0\right)=0\ \text{ and }\ \frac{\partial\theta\left(0\right)}{\partial S_i}=0\,,\forall\,i,j=1,2,3.$$

It follows from Eq. (53) that

$$\mathbb{T}_{ij}S_{ij}=2\left(\nu+\nu_1\left|\mathbf{S}\right|\right)\left|\mathbf{S}\right|^2\geq c_1\left(1+\left|\mathbf{S}\right|\right)\left|\mathbf{S}\right|^2 \tag{54}$$

and thus we find

$$\frac{d}{dt}\left|\mathbf{u}\right|_2^2+\left|\mathbf{u}\right|_2^2+\nu c_1C_2^2\left|\nabla\mathbf{u}\right|_2^2+\nu_1c_1C_3^3\leq\left\langle\mathbf{f},\mathbf{u}\right\rangle. \tag{55}$$

Now, applying Hölder's inequality, followed by Poincaré's inequality

$$\left|\mathbf{u}\right|_2\leq\lambda_1^{-\frac{1}{2}}\left|\nabla\mathbf{u}\right|_2^1$$

where λ_1 is the first eigenvalue of the Stokes operator, and using the following inequality

$$\lambda_1^{-\frac{1}{2}}\left|\mathbf{f}\right|_2\left|\nabla\mathbf{u}\right|_2\leq\frac{\nu c_1C_2^2}{2}\left|\nabla\mathbf{u}\right|_2^2+\frac{1}{2\nu c_1C_2^2\lambda}\left|\mathbf{f}\right|_2^2$$

we obtain

$$\frac{d}{dt}\left|\mathbf{u}\right|_2^2+\nu c_1C_2^2\left|\nabla\mathbf{u}\right|_2^2+2\nu_1c_1C_3^3\left|\nabla\mathbf{u}\right|_3^3\leq\frac{\left|\mathbf{f}\right|_2^2}{\nu c_1C_2^2\lambda_1}$$

respectively,

$$\frac{d}{dt}\left|\mathbf{u}\right|_2^2+\nu\lambda_1c_1C_2^2\left|\nabla\mathbf{u}\right|_2^2+2\nu_1c_1C_3^3\left|\nabla\mathbf{u}\right|_3^3\leq\frac{\left|\mathbf{f}\right|_2^2}{\nu c_1C_2^2\lambda_1} \tag{56}$$

The classical Gronwall lemma gives

$$\left|\mathbf{u}\right|_2^2\leq\left|\mathbf{u}_0\right|_2^2\exp\left(-\nu\lambda_1c_1C_2^2t\right)+\frac{\left|\mathbf{f}\right|_2^2}{\nu^2c_1^2C_2^4\lambda_1^2}\left(1-\exp\left(-\nu\lambda_1c_1C_2^2t\right)\right) \tag{57}$$

and thus we have

$$\lim_{t\to\infty}\text{sup}\left|\mathbf{u}\right|_2\leq\rho_0,\text{with}\,\rho_0=\frac{\left|\mathbf{f}\right|_2^2}{\nu c_1C_2^2\lambda_1}.$$

From (57) we infer that the balls of H of radius ρ are absorbing for all $\rho > \rho_0$. Indeed, let $\rho > \rho_0$ and denote by B_0 the ball $B_H(0, \rho)$.

Let B be any bounded set in H. Then, there exists $R > 0$ such that $B \subset B(0, R)$. Hence we have

$$|\mathbf{u}(t)|^2 \leq R^2 \exp\left(-\nu\lambda_1 c_1 C_2^2 t\right) + \rho_0^2 \left(1 - \exp\left(-\nu\lambda_1 c_1 C_2^2 t\right)\right). \tag{58}$$

It is obvious that the condition

$$R^2 \exp\left(-\nu\lambda_1 c_1 C_2^2 t\right) + \rho_0^2 \left(1 - \exp\left(-\nu\lambda_1 c_1 C_2^2 t\right)\right) < \rho^2$$

implies

$$S(t) B \subset B_0 \,, \forall t > t_0(B, \rho) = t_0 = \frac{1}{\nu\lambda_1 c_1 C_2^2} \log \frac{R^2}{\rho^2 - \rho_0^2}, \tag{59}$$

which proves that B_0 is an absorbing set in H.

□

CONCLUSIONS

This work brings significant scientific advances and refinements to the topic discussed in the work of (SANTOS; SALES, 2023). The main points of relevance are mentioned below:

a. **Refinement of the Smagorinsky Model:** The present work revisits the Smagorinsky model and offers a more rigorous mathematical analysis of its subgrid-scale formulation using the asymptotic analysis of the Large Eddy Simulation (LES) model. This contrasts with the work of (SANTOS; SALES, 2023), which mainly presents and applies the Smagorinsky model to turbulent flow problems, without a more theoretical in-depth, presented here.

b. **Dynamic Smagorinsky Model:** A significant contribution of the present work is the proposal of a dynamic Smagorinsky model where the C_S coefficient varies with space and time. This approach is suggested to improve model accuracy in representing various turbulent flows. The work of (SANTOS; SALES, 2023), on the other hand, treats the Smagorinsky constant C_S as a fixed value, which can lead to limitations in accurately capturing the dynamics of different turbulent flows.

c. **Error Minimization and Least-Square Method:** This work introduces a least squares method to minimize errors in the Smagorinsky dynamic model, specifically addressing the Germano identity and providing a detailed mathematical treatment of the error minimization process. This level of detail is not present in the work of (SANTOS; SALES, 2023), which does not explore the mathematical optimization of the Smagorinsky model parameters.

d. **Spectral Equilibrium Hypothesis:** This article discusses the local equilibrium hypothesis and its implications for maintaining a constant spectral equilibrium in turbulent flows. This concept ensures that energy does not accumulate at any frequency, keeping the shape of the energy spectrum unchanged over time. The article (SANTOS; SALES, 2023) does not address this hypothesis, focusing more on practical applications than on theoretical equilibrium states.

e. **Asymptotic Analysis and Anisotropic Viscosity:** This paper suggests that future research will likely lead to the development of a more in-depth anisotropic viscosity model for turbulent flow, which could resolve the regularity problem within the Navier-Stokes equations. This prospective advance is not mentioned in the article (SANTOS; SALES, 2023), indicating a perspective in this article, towards addressing fundamental issues in modeling turbulence with greater accuracy of physical reality.

Overall, this work provides a deeper and more nuanced exploration of the Smagorinsky model, introducing dynamical elements, rigorous mathematical analyses, and proposing future research directions that aim to resolve long-standing challenges in turbulence modeling. This represents a substantial scientific advance over the work of (SANTOS; SALES, 2023), which mainly presents a standard application of the Smagorinsky model.

In summary, this study rigorously re-examined the Smagorinsky model, shedding light on its subgrid-scale mathematical formulation through asymptotic analysis of the Large Eddy Simulation (LES) model. The elucidation provided by this mathematical analysis serves not only as a fundamental element, but also lays the foundation for a broader investigation into the regularity of the Navier-Stokes Equations. The author firmly believes that this investigation represents a significant step forward in advancing the Smagorinsky model, with the expectation that future research will culminate in a more detailed anisotropic viscosity model for turbulent flow, thus addressing the persistent issue of regularity within the Navier Equations. -Stokes. This effort aims to present a comprehensive mathematical analysis, encouraging further exploration and promoting a broader understanding of the enduring challenge posed by the regularity of the Navier-Stokes equations in one of the Millennium Problems.

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