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A note on Gaussian and Quaternion Repunit Numbers

Uma nota acerca dos números repunidades Gaussianos e Quaternions
Una nota sobre los números repunit de Gauss y Quaternion

Eudes Antonio Costa
<eudes@uft.edu.br>
Universidade Federal do Tocantins, Colegiado de Matemática, Arraias, To, Brasil
<https://orcid.org/0000-0001-6684-9961>

Douglas Catulio Santos
<catuliodouglas4@gmail.com>
Secretaria de Educação do Estado da Bahia, Barreiras, BA, Brasil
<https://orcid.org/0000-0002-5221-6087>

Paula Maria Machado Cruz Catarino
<pcatarin@utad.pt>
Universidade de Trás-os-Montes e Alto Douro (UTAD), Vila Real, Portugal
<https://orcid.org/0000-0001-6917-5093>

Elen Viviani Pereira Spreafico
<elen.spreafico@ufms.br>
Universidade Federal do Mato Grosso do Sul, Campo Grande, MS, Brasil
<https://orcid.org/0000-0001-6079-2458>

Abstract
This work introduces two new sequences: the gaussian repunit numbers and the quaternion repunit numbers. We establish some properties of these sequences, as well as, recurrence relations, the Binet formula, and Catalan's, Cassini's, and d'Ocganes identities.

Keywords: Repunit number. Horadam Gaussian and Quaternion numbers. Generating function
Mathematics Subject Classification 2020: 11B37, 11B39, 11B83 .

Resumo
Este trabalho introduz duas novas sequências: os números repunidades gaussianos e os números repunidades quaternions. Estabelecemos algumas propriedades dessas sequências, bem como relações de recorrência, a fórmula de Binet e as identidades de Catalan, Cassini e d'Ocganes.

Palavras-chave: *Número repunidade. Sequência de números Gassianos e Quaternios de Horadam. Funções generadoras*

Resumen
Este trabajo introduce dos nuevas secuencias: números de repunity gaussianos y números de repunity de cuaterniones. Establecemos algunas propiedades de estas secuencias, así como las relaciones de recurrencia, la fórmula de Binet y las identidades de Catalan, Cassini y d'Ocganes.

Palabras-Clave: *Número de repunity. Secuencia de cuaterniones gassianos y horadam. Funciones generadoras*

INTRODUCTION

Recursive relations define several families of integers that are studied in the literature. These sequences of numbers are at the origin of many interesting identities. One of these integer sequences is the repunit sequence $\{R_n\}_{n\geq 0}$ formed by integers numbers which are written in the decimal system as the repetition of the unit, and represented by the set $\{0, 1, 11, 111, \ldots\}$, sequence A002275 in the OEIS (SLOANE, 2024). The repunit sequence was introduced in (Yates, 1982), where the author was interested in answering the question: "Consider an integer N with n digits where each digit is a unit. For which values of n is N prime? "The connection with the repunit sequence, prime numbers, and the generalization of the repunit sequence was studied in (Beiler, 1964; Jaroma, 2007; Yates, 1982; Snyder, 1982). In (Snyder, 1982) was introduced the extended notion of a repunit to one in which for some integer $b > 1$, namely, $R_n(b) = \sum_{i=0}^{n-1} b^i = \dfrac{b^n - 1}{b - 1}$. The author gives a necessary and sufficient condition to $R_n(b)$ have a prime divisor congruent to 1 (mod n).

In (Jaroma, 2007) the author reproves the same result by using the theory of the Lucas sequences. For these sequences, the author defined the following recurrence sequence of integers

$$U_{n+2} = PU_{n+1} - QU_n \;, \forall\, n \geq 1, \tag{1}$$

where P and Q are any pair of relatively prime integers and $U_0 = 0$ and $U_1 = 1$. Similarly, in the same article, it is defined the companion Lucas sequence given by the following recurrence sequence of integers

$$V_{n+2} = PV_{n+1} - QV_n \;, \forall\, n \geq 1, \tag{2}$$

where P and Q are any pair of relatively prime integers and $U_0 = 2$ and $U_1 = P$. Observe that, the Lucas sequence considered in (Jaroma, 2007) is not the same as the classical Lucas sequence given by the sequence A000032 in OEIS (SLOANE, 2024), but if we consider $P = 1$ and $Q = -1$ in (2) we have that the classical Lucas numbers are a particular companion Lucas sequence (see, for instance, (Koshy, 2019; Vajda, 2008)).

Consider $P = 11$ and $Q = 10$ in Equation (1), then for all $n \geq 1$ we have the following recurrence

$$R_{n+1} = 11R_n - 10R_{n-1}, \tag{3}$$

with initial condition $R_0 = 0$ and $R_1 = 1$, which is nothing more than the repunit sequence of numbers. Since this sequence is given by a recurrence relation of order 2, we can provide the study of these numbers using the theory of recurrence sequences. In (Tarasov, 2007), the repunit numbers have been studied and interesting properties are proved. Recently, in (Santos; Costa, 2023), the authors introduced the recurrence (3) and explored the repunit sequence from this perspective. It is established some properties, such as the sum of n terms, as well as, several classical identities such as Catalan's and Cassini's identities.

In this work, we will generalize the repunit sequence concerned with the gaussian repunit and quaternion repunit numbers. Our main objective is to show the complexification process of the repunit sequence. In addition, we will establish properties of these new sequences, as well as, the Binet formula, the generating function, and several identities.

Horadam in (Horadam, 1963) introduced the concept of the complex Fibonacci numbers and the Fibonacci quaternion numbers, and established some quite general identities concerning them. From this work, many other classical sequences and generalizations were introduced in the complex, bicomplex, quaternion, and hybrid version, (see, for instance, (Diskaya; Menken, 2023; Harman, 1981; Horadam, 1963; Horadam, 1993; Iyer, 1969; Pethe; Horadam, 1986; Smith, 2004; Spreafico; Catarino; Vasco, 2023; Tasci, 2018; Vieira; Alves; Catarino, 2022)).

This complexification process is an extension of the integer sequence into a complex, quaternion, or hybrid set. It is important to note that, many of the results and properties remain in each set of numbers. It is possible to study the classical identities in the complex or quaternion set such as Cassini's, Catalan's, and d'Ocgnes identities, as well as, the Binet formula and generating function.

This paper is organized as follows. In Section 2, we discuss some preliminary results as the complex set, the quaternion set, and the definition of repunit numbers. In addition, we present the generating function for the repunit numbers that will be used in the next sections. In Sections 2 and 3 we present the definition of gaussian and quaternion repunit sequence, respectively, also properties, recurrence relation, generating function, sum formula, as well as Catalan's, Cassin's, and d'Ocgane's identities. In Section 3, for quaternions, it is interesting to note that, due to the non-commutativity of multiplication, there are two versions for each identity. Finally, some conclusions are stated.

1. BACKGROUND AND PRELIMINARIES RESULTS

Consider the field of complex numbers, denoted by $(\mathbb{C},+,\cdot)$. The set of complex numbers is defined as $\mathbb{C}=\{a+bi \ | \ a,\ b\in\mathbb{R} \text{ and } i^2=-1\}$, where i is complex unit. We know that $(\mathbb{C},+)$ and $(\mathbb{C},\cdot)$ are abelian groups, and then, the conjugate of a complex number $x=a+bi$ is defined by $\overline{x}=a-bi$. A gaussian number is a complex number $z=a+bi$, where a and b are integers, see (Felzenszwalb, 1979; Halici, 2012; Smith, 2004). For all $a,\ b,\ c,\ d$ integer numbers, the following arithmetic operations in the gaussian set holds:

$$(a+bi)^2=(a^2-b^2)+2abi\ ,$$
$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i\ .$$

The set of a quaternion (hamiltonian) numbers, denoted by $\mathbb{H}$, is defined as

$$\mathbb{H}=\{a+bi+cj+dk \mid a,\ b,\ c,\ d\in\mathbb{R} \text{ with } i^2=j^2=k^2=ijk=-1\}.$$

According to (Conway; Smith, 2003; Felzenszwalb, 1979; Messenger, 2014; Li *et al.*, 2009; Smith, 2004), $(\mathbb{H},+,\cdot)$ form a vector space with a base $1,\ i,\ j,\ k$, which is composed of unit 1 and its imaginary units $i,\ j$ and k. The addition of two quaternion numbers is defined by summing their components. So, the addition operation in the quaternion numbers is both commutative and associative. Zero is the null element. Concerning the addition operation, the symmetric element of x is $-x$, which is defined as having all the components of x changed in their signals. This implies that, $(\mathbb{H},+)$ is an abelian group. The conjugate of a quaternion number $x = a + bi + cj + dk$ is defined by $\overline{x} = a - bi - cj - dk$. When the real component is equal to zero, the quaternion is called pure. Quaternion multiplication follows the usual algebraic multiplication rules from the definition of quaternion numbers, the multiplication table of the quaternion units is given by Table 1:

TABELA 1.
The multiplication table for quaternion units

$\bullet$	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	$-k$	-1	i
k	k	j	$-i$	-1

In (Santos; Costa, 2023) the repunit sequence $\{R_n\}_{n\geq 0}$ is recursively defined in base 10 by recurrence (3), namely,

$$R_0 = 0 \; , R_1 = 1 \text{ and } R_{n+1} = 11R_n - 10R_{n-1} \; ,$$

where R_n denotes the n-th repunit number. The explicit formula for the $n-$th repunit number is given by

$$R_n = \frac{10^n - 1}{9} \; , \tag{4}$$

(see (Santos; Costa, 2023; Santos; Costa, 2024) and references therein). In addition, our next result presents a generating function for the repunit numbers.

Proposition 1. *The generating function for the repunit numbers $\{R_n\}_{n\geq 0}$ denoted by $G_{R_n}(x)$, is given by*

$$G_{R_n}(x) = \frac{x}{1 - 11x + 10x^2}. \tag{5}$$

Demonstração. Let $G_{R_n}(x) = \sum_{n=0}^{\infty} R_n x^n$ be the generating function of repunit numbers. Then, by expanding equations $G_{R_n}(x)$, $-11xG_{R_n}(x)$ and $10x^2 G_{R_n}(x)$, we obtain

$$\begin{aligned} G_{R_n}(x) =& R_0 + R_1 x + R_2 x^2 + \ldots + R_n x^n + \ldots \\ -11xG_{R_n}(x) =& -11R_0 x - 11R_1 x^2 - 11R_2 x^3 - \ldots - 11R_n x^{n+1} + \ldots \\ 10x^2 G_{R_n}(x) =& 10R_0 x^2 + 10R_1 x^3 + 10R_2 x^4 + \ldots + 10R_n x^{n+2} + \ldots . \end{aligned}$$

When we add to both sides, we have

$$\begin{aligned} (1 - 11x + 10x^2)G_{R_n}(x) =& R_0 + (R_1 - 11R_0)x + (R_2 - 11R_1 + 10R_0)x^2 \\ & + (R_3 - 11R_2 + 10R_1)x^3 + \ldots + (R_n - 11R_{n-1} \\ & + 10R_{n+2})x^n + \ldots \\ \overset{(3)}{=}& x + 0 \cdot x^2 + 0 \cdot x^3 + \ldots + 0 \cdot x^n + \ldots , \end{aligned}$$

which implies the result. □

2. GAUSSIAN REPUNIT NUMBERS

In this section, we will introduce the gaussian repunit numbers and provide some properties of these numbers. In addition, the Binet formula is provided, as well as several identities are established. Next, consider the definition of gaussian repunit numbers.

Definition 2. *For all integers $n \geq 0$, the n-th gaussian repunit number is defined by*

$$CR_n = R_n + R_{n+1}i, \tag{6}$$

where R_n is the n-th repunit number given by (3), $i^2 = -1$ and with initial conditions are $CR_0 = i$ and $CR_1 = 1 + 11i$.

As a consequence of Definition 2 and the recurrence relation (3) we have the following result.

Proposition 3. *The sequence $\{CR_n\}_{n\geq 0}$ of the gaussian repunit numbers satisfies the following second-order recursive relation:*

$$CR_{n+2} = 11CR_{n+1} - 10CR_n\ , \tag{7}$$

with initial conditions $CR_0 = i$ and $CR_1 = 1 + 11i$.

The recurrence $CR_{n+2} = 11CR_{n+1} - 10CR_n$ has characteristic equation given by

$$r^2 - 11r + 10 = 0\ , \tag{8}$$

whose roots are $r_1 = 10$ and $r_2 = 1$.

Let's determine the complex constants c_1 and c_2, considering that $CR_0 = i$ and $CR_1 = 1 + 11i$, and we obtain the system,

$$\begin{cases} i = c_1 + c_2 \\ 1 + 11i = 10c_1 + c_2\ . \end{cases}$$

Solving the system we find $c_1 = \dfrac{1+10i}{9}$ and $c_2 = -\dfrac{1+i}{9}$. Then, under the previous discussion, we can provide the Binet formula, as follows.

Proposition 4 (Binet's formula)**.** *For all $n \in \mathbb{N}$, we have*

$$CR_n = \frac{10^n - 1}{9} + \frac{10^{n+1} - 1}{9}i\ . \tag{9}$$

Demonstração. We have that a general solution to Equation (7) is of the form $CR_n = c_1(10)^n + c_2(1)^n$. Then, we obtain

$$\begin{aligned} CR_n &= c_1(10)^n + c_2(1)^n \\ &= \frac{1+10i}{9}(10)^n - \frac{1+i}{9}(1)^n \\ &= \frac{10^n - 1}{9} + \frac{10^{n+1} - 1}{9}i\ . \end{aligned}$$

□

Now, we consider the sequence of partial sums $CS_n = CR_0 + CR_1 + CR_2 + \cdots + CR_n$, for $n \geq 0$, where $\{CR_n\}_{n\geq 0}$ is the gaussian repunit sequence.

Proposition 5. *Let $\{CR_n\}_{n\geq 0}$ be the gaussian repunit sequence, then*

$$CS_n = \frac{10^{n+1} - 10 - 9n}{81} + \frac{10^{n+2} - 19 - 9n}{81} i \,.$$

Demonstração. We have that

$$\begin{aligned} CS_n &= CR_0 + CR_1 + \cdots + CR_n \\ &= (R_0 + R_1 + \cdots + R_n) + (R_1 + R_2 + \cdots + R_{n+1})i \\ &= (R_1 + \cdots + R_n) + (R_1 + R_2 + \cdots + R_{n+1})i \,. \end{aligned}$$

It is follows from Proposition 9 in (Santos; Costa, 2023) that

$$R_1 + R_2 + \cdots + R_n = S_n = \frac{10(10^n - 1) - 9n}{81} \,.$$

Then, we have

$$CS_n = \frac{10(10^n - 1) - 9n}{81} + \frac{10(10^{n+1} - 1) - 9(n+1)}{81} i \,.$$

□

Now, we will establish the classical Catalan's identity.

Proposition 6 (Catalan's Identity)**.** *Let m, n be any natural. For $m \geq n$ we have*

$$(CR_m)^2 - CR_{m-n}CR_{m+n} = -9 \cdot 10^{m-n} \cdot (R_n)^2 + \big(10^{m-n}(10^n + 1)R_n\big)\, i \,.$$

Demonstração. It is follows from Equation (6)

$$\begin{aligned} &(CR_m)^2 - CR_{m-n}CR_{m+n} \\ =&(R_m + R_{m+1}i)^2 - (R_{m-n} + R_{(m-n)+1}i)(R_{m+n} + R_{(m+n)+1}i) \\ =&\big((R_m^2 - R_{m+1}^2) + 2R_mR_{m+1}i\big) \\ &- \big((R_{m-n}R_{m+n} - R_{(m-n)+1}R_{(m+n)+1}) + (R_{m-n}R_{(m+n)+1} + R_{(m-n)+1}R_{m+n})i\big) \\ =&(R_m^2 - R_{m-n}R_{m+n}) - (R_{m+1}^2 - R_{(m+1)-n}R_{(m+1)+n}) \\ &+ (2R_mR_{m+1} - R_{m-n}R_{(m+n)+1} - R_{(m-n)+1}R_{m+n})i \,. \end{aligned}$$

In real component, by applying Proposition 6 in (Santos; Costa, 2023), we have

$$\begin{aligned} &(R_m^2 - R_{m-n}R_{m+n}) - (R_{m+1}^2 - R_{(m+1)-n}R_{(m+1)+n}) \\ &= 10^{m-n} \cdot (R_n)^2 - 10^{(m+1)-n} \cdot (R_n)^2 \\ &= -9 \cdot 10^{m-n} \cdot (R_n)^2 \,. \end{aligned} \tag{10}$$

Now, by replacing Binet formula, Equation (4), in complex component, we obtain that

$$\begin{aligned}
&2R_mR_{m+1} - R_{m-n}R_{(m+n)+1} - R_{(m-n)+1}R_{m+n} \\
=&2\left(\frac{10^m-1}{9}\right)\left(\frac{10^{m+1}-1}{9}\right) - \left(\frac{10^{m-n}-1}{9}\right)\left(\frac{10^{(m+n)+1}-1}{9}\right) \\
&-\left(\frac{10^{(m-n)+1}-1}{9}\right)\left(\frac{10^{m+n}-1}{9}\right) \\
=&\frac{10^{(m+n)+1}+10^{m+n}+10^{(m-n)+1}+10^{m-n}-2\cdot 10^{m+1}-2\cdot 10^m}{81} \\
=&\frac{10^{m+n}+10^{m-n}-2\cdot 10^m}{9} = \frac{10^m(10^n-1)}{9} + \frac{10^{m-n}(10^n-1)}{9} \\
=&10^{m-n}(10^n+1)R_n\,. \qquad (11)
\end{aligned}$$

By adding Equations (10) and (11) we have the result. □

For $n=1$, the Cassini identity below follows directly from Proposition 6.

Corollary 7. *[Cassini's Identity] For all $m \geq 1$, we have*

$$(CR_m)^2 - CR_{m-1}CR_{m+1} \;=\; -9\cdot 10^{m-1} + 11\cdot 10^{m-1}i\,.$$

Similar to Proposition 6, we can obtain the d'Ocgane's identity, as follows.

Proposition 8 (d'Ocagne's Identity)**.** *Let m,n be any natural. For $m \geq n$ we have*

$$CR_mCR_{n+1} - CR_{m+1}CR_n \;=\; -9\cdot 10^n R_{m-n} - 9\cdot 10^{n+1}R_{m-n}i\,.$$

Demonstração. Using Equation (6) we obtain that

$$\begin{aligned}
&CR_mCR_{n+1} - CR_{m+1}CR_n \\
=&(R_m + R_{m+1}i)(R_{n+1}+R_{n+2}i) - (R_{m+1}+R_{m+2}i)(R_n + R_{n+1}\mathbf{i}) \\
=&\left((R_mR_{n+1} - R_{m+1}R_{n+2}) + (R_mR_{n+2} + R_{m+1}R_{n+1})i\right) \\
&-\left((R_{m+1}R_n - R_{m+2}R_{n+1}) + (R_{m+1}R_{n+1} + R_{m+2}R_n)i\right) \\
=&\left((R_mR_{n+1} - R_{m+1}R_n) - (R_{m+1}R_{n+2} - R_{m+1}R_{n+1})\right) \qquad (12) \\
&+\left((R_mR_{n+2} - R_{m+1}R_{n+1}) - (R_{m+1}R_{n+2} - R_{m+2}R_{n+1})\right)i \qquad (13)
\end{aligned}$$

By applying Proposition 5 in (Santos; Costa, 2023) in Equation (12), we have

$$\begin{aligned}
&(R_mR_{n+1} - R_{m+1}R_n) - (R_{m+1}R_{n+2} - R_{m+1}R_{n+1}) \\
=&10^nR_{m-n} - 10^{n+1}R_{m+1-(n+1)} \\
=&-9\cdot 10^nR_{m-n}\,; \qquad (14)
\end{aligned}$$

Similarly, by direct application of Proposition 5 in (Santos; Costa, 2023) in Equation 13 we obtain

$$\begin{aligned}
&(R_mR_{n+2} - R_{m+1}R_{n+1}) - (R_{m+1}R_{n+2} - R_{m+2}R_{n+1}) \\
=&10^{n+1}R_{m-(n+1)} - 10^{n+1}R_{m+1-(n+1)} \\
=&-9\cdot 10^{n+1}R_{m-n}\,. \qquad (15)
\end{aligned}$$

Thus, combining Equations (14) and (15), the result is verified. □

Next, we will present the generating function for the gaussian repunit sequence.

Proposition 9. *The generating function for the gaussian repunit sequence* $\{CR_n\}_{n\geq 0}$, *denote by* $G_{CR_n}(x)$, *is*

$$G_{CR_n}(x) = \frac{x}{1-11x+10x^2} + \frac{1}{1-11x+10x^2}i\ .$$

Demonstração. Let $G_{CR_n}(x) = \sum_{n=0}^{\infty} CR_n x^n$ be the generating function for the gaussian repunit sequence. Combining the expressions $-11xG_{CR_n}(x)$ and $10x^2 G_{CR_n}(x)$, we have

$$\begin{aligned} G_{CR_n} =& \frac{CR_0 + (CR_1 - 11CR_0)x}{1-11x+10x^2} \\ =& \frac{i + (1 + 11i - 11i)}{1-11x+10x^2} \\ =& \frac{x}{1-11x+10x^2} + \frac{1}{1-11x+10x^2}i\ , \end{aligned}$$

which verifies the result. □

3. QUATERNION REPUNIT NUMBERS

Similarly to the previous section, now, we will present the quaternion repunit numbers and provide some properties, as well as, a sum formula, the Catalan, the Cassini, and d'Ocgane's identities.

Consider the following definition of the quaternion repunit sequence.

Definition 10. *For all integers* $n \geq 0$, *the set of quaternion repunit numbers is denoted by* $\{HR_n\}_{n\geq 0}$ *and defined by*

$$HR_n = R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k\ , \tag{16}$$

where R_n *is the* n*-th repunit number, with initial conditions* $HR_0 = i + 11j + 111k$ *and* $HR_1 = 1 + 11i + 111j + 1111k$.

The following proposition establishes the Horadam recurrence relation for the quaternion repunit numbers.

Proposition 11. *For all* $n \geq 0$, *the quaternion repunit sequence* $\{HR_n\}_{n\geq 0}$ *satisfy the recurrence relation*

$$HR_{n+2} = 11HR_{n+1} - 10HR_n \tag{17}$$

Demonstração. It follows from Equation (16)

$$\begin{aligned} & 11HR_{n+1} - 10HR_n \\ =& 11(R_{n+1} + R_{n+2}i + R_{n+3}j + R_{n+4}k) - 10(R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k) \\ =& (11R_{n+1} - 10R_n) + (11R_{n+2} - 10R_{n+1})i + (11R_{n+3} - 10R_{n+2})j \\ & + (11R_{n+4} - 10R_{n+3})k \\ =& R_{n+2} + R_{n+3}i + R_{n+4}j + R_{n+5}k \\ =& HR_{n+2}\ . \end{aligned}$$

□

The next result provides the Binet formula for quaternions repunit numbers.

Proposition 12. *[Binet's Formula] For all $n \in \mathbb{N}$, we have*

$$HR_n = \frac{10^n - 1}{9} + \frac{10^{n+1} - 1}{9}i + \frac{10^{n+2} - 1}{9}j + \frac{10^{n+3} - 1}{9}k \tag{18}$$

Demonstração. We have that a general solution to (17) is in the form $HR_n = c_1(10)^n + c_2(1)^n$. Let's determine the quartenions constants c_1 and c_2, considering $HR_0 = i + 11j + 111k$ and $HR_1 = 1 + 11i + 111j + 1111k$. Thus, we obtain the linear system

$$\begin{cases} i + 11j + 111k = c_1 + c_2 \\ 1 + 11i + 111j + 1111k = 10c_1 + c_2 \, . \end{cases}$$

Solving the linear system we find $c_1 = \dfrac{1 + 10i + 100j + 1000k}{9}$ and $c_2 = -\dfrac{1 + 1i + 1j + 1k}{9}$. Therefore, the particular solution is given by

$$HR_n = \frac{(1 + 10i + 100j + 1000k)10^n - (1 + i + j + k)}{9} \, .$$

□

The next result establishes formula for the sum of the first n terms o HR_n, proceeding similarly to what was done with complex one. Consider the sequence of partial sums

$$HS_n = HR_0 + HR_1 + HR_2 + HR_3 + \ldots + HR_n \, ,$$

then we have the following result.

Proposition 13. *Let $(HR_n)_{n \geq 0}$ be the quaternion repunit sequence, then*

$$HS_n = \frac{10^{n+1} - 10 - 9n}{81} + \frac{10^{n+2} - 19 - 9n}{81}i + \frac{10^{n+3} - 109 - 9n}{81}j + \frac{10^{n+4} - 1009 - 9n}{81}k \, .$$

Demonstração. We have that

$$\begin{aligned} HS_n =& HR_0 + HR_1 + HR_2 + HR_3 + \ldots + HR_n \\ =& (R_1 + R_2 + \ldots + R_n) + (R_1 + R_2 + \ldots + R_{n+1})i \\ & + (R_2 + R_3 + \ldots + R_{n+2})j + (R_3 + R_4 + \ldots + R_{n+3})k \\ =& (R_1 + R_2 + \ldots + R_n) + (R_1 + R_2 + \ldots + R_{n+1})i \\ & + (R_1 + R_2 + R_3 + \ldots + R_{n+2})j \\ & + (R_1 + R_2 + R_3 + R_4 + \ldots + R_{n+3})k - (j + 12k) \end{aligned}$$

From Proposition 9 in (Santos; Costa, 2023) it is follows that

$$\begin{aligned} HS_n =& \frac{10^{n+1} - 10 - 9n}{81} + \frac{10^{n+2} - 19 - 9n}{81}i \\ & + \frac{10^{n+3} - 28 - 9n}{81}j + \frac{10^{n+4} - 37 - 9n}{81}k - (j + 12k) \, . \end{aligned}$$

□

The lemma below is an auxiliary result that plays an important role in Catalan's identity.

Lemma 14. *For all* $a, b, c, d \in \mathbb{R}$ *and* $A = 1 + i + j + k \in \mathbb{Q}$ *then:*

$$\begin{aligned}
&(1)(a + bi + cj + dk)A \\
&=(a - b - c - d) + (a + b + c - d)i + (a - b + c + d)j + (a + b - c + d)k\,. \qquad (19)\\
&(2)\ A(a + bi + cj + dk) \\
&=(a - b - c - d) + (a + b - c + d)i + (a + b + c - d)j + (a - b + c + d)k\,. \qquad (20)\\
&(3)\ A^2 = -2 + 2i + 2j + 2k\,. \qquad (21)
\end{aligned}$$

As a consequence of Lemma (14) we have

Lemma 15. *For* $A = 1 + i + j + k$ *and* $B = 1 + 10i + 100j + 1000k \in \mathbb{Q}$ *then:*

$$\begin{aligned}
&(1)\ BA = -1109 - 889i + 1091j + 911k\,.\\
&(2)\ AB = -1109 + 911i - 889j + 1091k\,.
\end{aligned}$$

Demonstração. (1) Indeed, by Lemma (14), we have that

$$\begin{aligned}
BA =&(1 - 10 - 100 - 1000) + (1 + 10 + 100 - 1000)i + (1 - 10 + 100 + 1000)j\\
&+ (1 + 10 - 100 + 1000)k\\
=& - 1109 - 889i + 1091j + 911k\,.
\end{aligned}$$

(2) Once again, by Lemma (14), we have that

$$\begin{aligned}
AB =&(1 - 10 - 100 - 1000) + (1 + 10 - 100 - 1000)i + (1 + 10 + 100 - 1000)j\\
&+ (1 - 10 + 100 + 1000)k\\
=& - 1109 + 911i - 889j + 1091k\,.
\end{aligned}$$

□

Now, we present the classical identities associated with the quaternion repunit sequence. Recall that the multiplication is not commutative in quaternion sets, then for each identity, we will have two versions, that we call the first and second identities. We begin with the first Catalan identity.

Proposition 16. *[First Catalan's Identity] For all* $m,\ n \in \mathbb{N}$. *If* $m \geq n$ *then*

$$HR_m^2 - HR_{m-n}HR_{m+n} = \frac{10^{m-n}}{9}R_n$$
$$\left[-1109R_n + \left(\frac{911 \cdot 10^n + 889}{9}\right)i - \left(\frac{889 \cdot 10^n + 1091}{9}\right)j + \left(\frac{1091 \cdot 10^n + 911}{9}\right)k\right]\,.$$

Demonstração. Note that, by Proposition 12, it follows that

$$HR_m = \frac{(1 + 10i + 100j + 1000k)10^m - (1 + i + j + k)}{9} = \frac{B10^m - A}{9}\,,$$

consider $B = (1 + 10i + 100j + 1000k)$ and $A = (1 + i + j + k)$, then

$$
\begin{aligned}
&HR_m^2 - HR_{m-n}HR_{m+n} \\
=&\left(\frac{B10^m - A}{9}\right)\left(\frac{B10^m - A}{9}\right) - \left(\frac{B10^{n-m} - A}{9}\right)\left(\frac{B10^{n+m} - A}{9}\right) \\
=&\left(\frac{B^2 10^{2m} - BA10^m - AB10^m + A^2}{9^2}\right) \\
&-\left(\frac{B^2 10^{2m} - BA10^{m-n} - AB10^{m+n} - A^2}{9^2}\right) \\
=&\frac{AB10^m(10^n - 1)}{9^2} - \frac{BA10^{m-n}(10^n - 1)}{9^2} \\
=&\frac{10^n - 1}{9}\left(\frac{AB10^m - BA10^{m-n}}{9}\right) \\
=&R_n\left(\frac{AB10^m - BA10^{m-n}}{9}\right) \\
=&R_n\left(\frac{10^{m-n}(AB10^n - BA)}{9}\right) \\
=&\frac{10^{m-n}}{9}R_n \cdot (AB10^n - BA)\,.
\end{aligned}
$$

Now, by applying Lemma 15 in $AB10^n - BA$, implies that

$$
\begin{aligned}
&HR_m^2 - HR_{m-n}HR_{m+n} = 10^{m-n}R_n \\
&\left[-1109R_n + \left(\frac{911 \cdot 10^n + 889}{9}\right)i - \left(\frac{889 \cdot 10^n + 1091}{9}\right)j + \left(\frac{1091 \cdot 10^n + 911}{9}\right)k\right]\,.
\end{aligned}
\tag{22}
$$

□

The Cassini identity follows directly from Proposition 16, given by the following corollary.

Corollary 17. *[First Cassini's Identity] For all $m \geq 1$, we have*

$$
\begin{aligned}
&HR_m^2 - HR_{m-1}HR_{m+1} \\
=&10^{m-1}\left[-1109 + 1111i - 1109j + \left(\frac{11821}{9}\right)k\right]\,.
\end{aligned}
$$

Demonstração. The result is verified by doing $n = 1$ in Equation (22). □

The next result consists of the first version of d'Ocagne's identity.

Proposition 18. *[First d'Ocagne's identity] Let m, n be natural. For $m \geq n$ we have*

$$
\begin{aligned}
&HR_{m+1}HR_n - HR_mHR_{n+1} = \frac{10^n}{9} \\
&= \left[-9171R_{m-n} - \left(889 \cdot 10^{m-n} + 911\right)\mathbf{i} + \left(1091 \cdot 10^{m-n} + 889\right)\mathbf{j} + \left(911 \cdot 10^{m-n} - 1091\right)\mathbf{k}\right]\,.
\end{aligned}
$$

Demonstração. Again, by Proposition 12, we have that

$$HR_m = \frac{(1+10i+100j+1000k)10^m - (1+i+j+k)}{9} = \frac{B10^m - A}{9},$$

consider $B = (1+10i+100j+1000k)$ and $A = (1+i+j+k)$, then

$$\begin{aligned} & HR_m HR_{n+1} - HR_{m+1} HR_n \\ =& \left(\frac{B10^m - A}{9}\right)\left(\frac{B10^{n+1} - A}{9}\right) - \left(\frac{B10^{m+1} - A}{9}\right)\left(\frac{B10^n - A}{9}\right) \\ =& \left(\frac{B^2 10^{m+n+1} - BA10^m - AB10^{n+1} + A^2}{9^2}\right) \\ & - \left(\frac{B^2 10^{m+n+1} - BA10^{m+1} - AB10^n - A^2}{9^2}\right) \\ =& \frac{BA10^m(10-1)}{9^2} - \frac{AB10^n(10-1)}{9^2} \\ =& \frac{10^n}{9}(BA10^{m-n} - AB). \end{aligned}$$

Now, by applying Lemma 15 in $(BA10^{m-n} - AB)$, we conclude that

$$HR_m HR_{n+1} - HR_{m+1} HR_n = 10^n \left[-1019 R_{m-n} - \left(\frac{889 \cdot 10^{m-n} + 911}{9}\right) i + \left(\frac{1091 \cdot 10^{m-n} + 889}{9}\right) j + \left(\frac{911 \cdot 10^{m-n} - 1091}{9}\right) k\right].$$

□

Since multiplication in $\mathbb{H}$ is not commutative, we also have similar results.

Proposition 19. *[Second Catalan's Identity] For all $m,\ n \in \mathbb{N}$. If $m \geq n$ then*

$$HR_m^2 - HR_{m+n} HR_{m-n} = 10^{m-n} R_n \left[-1109 R_n - \left(\frac{889 \cdot 10^n + 911}{9}\right) i + \left(\frac{1091 \cdot 10^n + 889}{9}\right) j + \left(\frac{911 \cdot 10^n - 1091}{9}\right) k\right].$$

Follows directly from Proposition 19 the second Cassini's identity.

Corollary 20. *[Second Cassini's Identity] For all $m \geq 1$, we have*

$$\begin{aligned} & HR_m^2 - HR_{m+1} HR_{m-1} \\ =& 10^{m-1}\left[-1109 - \left(\frac{9881}{9}\right) i + 1311 j + 891 k\right]. \end{aligned}$$

Proposition 21. *[Second D'Ocagne's Identity] Let m, n be natural. For $m \geq n$ we have*

$$HR_n HR_{m+1} - HR_{n+1} HR_m = \frac{10^n}{9} \left[-1109 R_{m-n} + \left(\frac{911 \cdot 10^{m-n} + 889}{9}\right) i - \left(\frac{889 \cdot 10^{m-n} + 1091}{9}\right) j + \left(\frac{1091 \cdot 10^{m-n} + 911}{9}\right) k\right].$$

The next result is the generating function for the quaternion repunit numbers.

Proposition 22. *The generating function for the quaternion repunit numbers is given by*

$$G_{HR_n}(x) = \frac{x}{1-11x+10x^2} + \frac{1}{1-11x+10x^2}i + \frac{11-10x}{1-11x+10x^2}j + \frac{111-110x}{1-11x+10x^2}k.$$

Demonstração. Consider the generating function for the sequence of quaternion repunit numbers given by $G_{HR_n}(x) = \sum_{n=0}^{\infty} Q_n x^n$. Combining expressions $-11xG_{HR_n}(x)$ e $10x^2 G_{HR_n}(x)$, we obtain

$$\begin{aligned} G_{HR_n}(x) =& \frac{HR_0 + (11HR_1 - 11HR_0)x}{1-11x+10x^2} \\ =& \frac{i + 11j + 111k + (1 + 11i + 111j + 1111k - 11i - 121j - 1221k)}{1-11x+10x^2} \\ =& \frac{x}{1-11x+10x^2} + \frac{1}{1-11x+10x^2}i + \frac{11-10x}{1-11x+10x^2}j + \frac{111-110x}{1-11x+10x^2}k\ , \end{aligned}$$

which concludes the proof. □

4. CONSIDERATIONS

In this paper, we introduced two new application of the Horadam sequences: the gaussian repunit number and the quaternion repunit number. In addition, we presented the generating function for the repunit numbers. We provided some properties, recurrence relation, generating function, and sum formula, as well as the Catalan, the Cassini, and the d'Ocgane identities for each new sequence. In particular, for quaternions, due to the non-commutativity of multiplication, there are two versions for each identity.

As far as we know, the results presented here are new in the literature. Moreover, the repunit numbers can be studied in another set of numbers, and also in other perspectives, such as matrix and combinatorial.

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