RMAT ISSN: 2237-8103
REVISTA DE MATEMÁTICA DA UFOP

OPEN ACCESS
Volume 1, Número 1, 2025
https://doi.org/10.63801/rmat.v1i.7773
Recebido: xx
Publicado: xx

Triquaternion ring: the tricomplex ring with complex coefficients

Anel Triquaternion: o anel tricomplexo com coeficientes complexos
Anillo Triquaterinion: el anillo tricomplejo con coeficientes complejos

Eudes Antonio Costa
<eudes@uft.edu.br>
Universidade Federal do Tocantins, Colegiado de Matemática, Arraias, To, Brasil
<https://orcid.org/0000-0001-6684-9961>
Keidna Cristiane Oliveira Souza
<keidna@uft.edu.br>
Universidade Federal do Tocantins, Campus Palmas, To, Brasil
<https://orcid.org/0000-0001-8404-7380>

Abstract
This study aims to explore and develop results related to the fundamental law of arithmetic within the framework of a commutative ring with unity. Specifically, it focuses on extending complex numbers to a vector space characterized by three complex coordinates, bridging foundational theoretical concepts with practical applications. Considering the extension of integer number sequences into other numerical sets, this research investigates a novel set of numbers. The extension of real numbers to higher dimensions, such as quaternions and octonions, has gained significance in physics due to their natural representation of certain symmetries in physical systems. In this work, we illustrate how the properties of complex numbers can be systematically leveraged to derive both the foundational basis and the multiplication rules for these advanced numerical systems.

Keywords: *Integer number. Complex number. Tricomplex number. Complex-Tricomplex number.*

Resumo
Este estudo tem como objetivo explorar e desenvolver resultados relacionados com a lei fundamental da aritmética no âmbito de um anel comutativo com unidade. Especificamente, centra-se na extensão dos números complexos a um espaço vetorial caracterizado por três coordenadas complexas, fazendo a ponte entre conceitos teóricos fundamentais e aplicações práticas. Considerando a extensão de sequências de números inteiros a outros conjuntos numéricos, este trabalho investiga um novo conjunto de números. A extensão dos números reais a dimensões superiores, como os quaterniões e os octónios, ganhou importância na física devido à sua representação natural de certas simetrias em sistemas físicos. Neste trabalho, ilustramos como as propriedades dos números complexos podem ser sistematicamente aproveitadas para derivar tanto a base fundamental como as regras de multiplicação para estes sistemas numéricos avançados.

Palavras-chave: *Número inteiro. Número complexo. Número Tricomplexo. Número complexo-Tricomplexo.*

Resumen
El objetivo de este estudio es explorar y desarrollar resultados relacionados con la ley fundamental de la aritmética en el contexto de un anillo conmutativo con unidad. En concreto, se centra en la extensión de los números complejos a un espacio vectorial caracterizado por tres coordenadas complejas, tendiendo un puente entre los conceptos teóricos fundamentales y las aplicaciones prácticas. Considerando la extensión de secuencias de números enteros a otros conjuntos numéricos, esta investigación indaga en un nuevo conjunto de números. La extensión de los números reales a dimensiones superiores, como los cuaterniones y los octoniones, ha cobrado importancia en física debido a su representación natural de ciertas simetrías en los sistemas físicos. En este trabajo, ilustramos

cómo pueden utilizarse sistemáticamente las propiedades de los números complejos para derivar tanto la base fundamental como las reglas de multiplicación de estos sistemas numéricos avanzados.

Palabras-Clave: *Número entero. Número complejo. Número tricomplejo. Número complejo-tricomplejo.*
MSC2020 – 13B02, 11R04, 01A65

1. INTRODUCTION

This paper introduces a new ring, the Tricomplex ring with a complex component embedded in $\mathbb{C}$. We will examine intricate structures in complex vector spaces from an algebraic perspective. The main result establishes the existence of a commutative ring with unity in a three-dimensional space over the field of complex numbers. This result extends the findings proposed by Olariu (Olariu, 2002) and (Olariu, 2000) for the field of real numbers. Additionally, a detailed description of these structures is provided. There is a noticeable lack of literature offering complete demonstrations of such structures. Classical textbooks on algebraic structures or linear algebra typically focus on, or are confined to, the classical sets of natural, integer, and rational numbers. Few courses address structures beyond real or complex numbers, and one of the objectives of this article is to contribute to filling this gap.

Since the work of Gauss, several researchers have been interested in studying rings that have arithmetic similar to that of integers. These include the Gauss ring of integers, which consists of complex numbers whose real and imaginary parts are integers. Another example of great importance in this context is the Eisenstein ring of integers. This paper presents a ring in three complex coordinates with complex components, focusing on its algebraic properties.

Several studies have focused on different number systems, with particular emphasis on complex numbers and their generalizations. These quaternions, octonions and hybrid numbers represent an extension or generalization of complex numbers and have attracted considerable attention from researchers. In the history of Mathematics, many researchers have been interested in studying rings analogous to the ring of integers, in which arithmetic concepts can also be developed. Among them, we highlight the ring of Gaussian integers, whose study originates from Gauss's investigations regarding cubic and biquadratic reciprocity. What makes this ring interesting is the fact that many arithmetic results in Gaussian integers are analogous to the results of arithmetic results in integers and can be illustrated geometrically. In this paper, we define the *Complex-Tricomplex* numbers.

The structure of this work is as follows: In Section 2, we provide a brief introduction to numerical sets beyond the familiar natural, integer, rational, and real numbers. This aims to spark curiosity or motivate the question: Is there a numerical set analogous to the integers within three-dimensional space over the field of complex numbers? In Section 3, we define the set of Complex-Tricomplex elements and demonstrate that it forms a unitary commutative ring embedded in the space $\mathbb{C}^3$.

2. BACKGROUND

In this section we take up some essential concepts about the field of complex numbers $\mathbb{C}$ and the ring *Real-Tricomplex*, or just *Tricomplex* $\mathbb{T} = \mathbb{TC}$. This basis will be fundamental for the subsequent development, which consists of showing how the set *Complex-Tricomplex* is also a ring with the same operations of addition and multiplication defined in $\mathbb{TQ}$, which we call here *Triquaternion*. The purpose of these notes is to provide an introduction to the Theory of Associative Algebras. This topic in algebra is significant not only in its own right but also for its connections to other areas of mathematics, physics, and genetics.

The construction or historical facts about the structure of the complex number field can be consulted in (Baumgart, 1992; Boyer; Merzbach, 2011; Costa; Bastos, 2012; Eves, 2008; Felzenszwalb, 1979; Hefez; Villela, 2012; Lubeck, 2024; Milies, 2004; Monteiro, 1969; Santos, 2023; Roque, 2012; Roque; Carvalho, 2012), among others. The formula that provides the roots of the equation $ax^2 + bx + c = 0$ in terms of the radical $\sqrt{b^2 - 4ac}$ was known in antiquity by the Babylonians. During the Italian Renaissance (15th and 16th centuries), considerable effort was dedicated to generalizing the method of solving equations using radicals. The work *Ars Magna*, published by Cardano in 1545, contains the method for solving cubic equations developed by Scipione del Ferro and Tartaglia, as well as the method for solving quartic equations developed by Ferrari.

The attempt to solve an equation often led to square roots of negative numbers, and when this occurred, it was considered that the equation had no solution. In 1572, Bombelli observed that radicals such as $\sqrt{-1}$ had no meaning, but it was possible to perform calculations with these radicals when they canceled out. The development of calculations involving complex numbers progressed, despite significant doubts about their validity, as they were referred to as *imaginary* or *impossible* numbers. The graphical representation of these numbers as vectors or points in the plane was mentioned by Wallis (1673), Wessel (1797), Argand (1813), Warren (1828), and Gauss (1832).

Initially, the symbolic approach to algebra was rejected by the Irish mathematician Hamilton. However, in 1833, Hamilton proposed a reformulation of the complex numbers, representing the complex number $a + bi$ in the form of the ordered pair (a, b) of real numbers. The mystery surrounding the roots of the equation $x^2 + 1 = 0$ disappears, as its roots are $(0, 1)$ and $(0, -1)$. The expressions

$$(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

define a field structure in $\mathbb{R}^2$, with a, b, c and d real numbers.

Hamilton devoted a significant portion of his career to investigating the potential for defining a multiplication in three-dimensional Euclidean space that would also result in a field. This endeavor ultimately proved to be unattainable. In 1843, Hamilton finally succeeded in defining a multiplication in $\mathbb{R}^4$, where: The following equations are true:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j,$$

with $1 = (1, 0, 0, 0)$, $i = (0, 1, 0, 0)$, $j = (0, 0, 1, 0)$, and $k = (0, 0, 0, 1)$. Hamilton demonstrated that all the axioms of the field are satisfied, with the exception of commutativity. This number system, designated as *quaternions*, constituted the inaugural example of a non-commutative field. Some months after the discovery of quaternions, Graves introduced the *octonions*, defining a multiplication in $\mathbb{R}^8$. This system of numbers was independently discovered by Cayley in 1845 and is therefore also referred to as *Cayley numbers*. This number system is neither commutative nor associative.

From this point, various other examples and generalizations emerged. In his work *Lecture on Quaternions* (1853), Hamilton introduced the concept of *hypercomplex numbers*. A hypercomplex number system consists of all symbols of the form:

$$x_1e_1 + x_2e_2 + \cdots + x_ne_n,$$

where $x_1, x_2, \ldots, x_n$ are real or complex numbers, and $e_1, e_2, \ldots, e_n$ are the units of the system. In the article *A Memoir On The Theory of Matrices*, published in 1858, Cayley introduced what is now known as the *algebra of matrices*. Other contributions were made by Grassmann (1844), Clifford (1878), and Sylvester (1884). The classification of associative hypercomplex number systems, that is, associative algebras, was extensively studied during the second half of the 19th century by B. Peirce and his son C. S. Peirce, Frobenius, Scheffers, Molien, Cartan, and others. The results obtained formed a satisfactory theory for algebras over $\mathbb{R}$ or $\mathbb{C}$. This structural theory was extended to finite-dimensional algebras over an arbitrary field by Wedderburn in 1907. In his article *On Hypercomplex Numbers*, Wedderburn included a section on non-associative algebras.

Here, we consider that a complex number $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}; i^2 = -1\}$ can be expressed as $\alpha = a + bi$ where a is the real part and b is the imaginary part, with i being the imaginary unit, where $i^2 = -1$. In $\mathbb{C}$ we consider the two operations: addition and multiplication. So the addition of two complex numbers $\alpha_1 = a_1 + b_1 i$ and $\alpha_2 = a_2 + b_2 i$ is given by:

$$\alpha_1 + \alpha_2 = (a_1 + a_2) + (b_1 + b_2)i \ ,$$

and the multiplication of two complex numbers $\alpha_1 = a_1 + b_1 i$ and $\alpha_2 = a_2 + b_2 i$ by:

$$\alpha_1 \cdot \alpha_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i \ ;$$

where $a_i, b_i \in \mathbb{R}$ for $i = 1, 2$. For all $\alpha \in \mathbb{C}$ the complex conjugate of $\alpha = a + bi$ is $\overline{\alpha} = a - bi$.

Other set of numbers was introduced by Özdemir in (Özdemir, 2018), where the author defines the set of hybrid numbers, which includes complex, dual, and hyperbolic numbers. This set is given by

$$\mathbb{K} = \{a + bi + c\epsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, i^2 = -1, \epsilon^2 = 0, \mathbf{h}^2 = 1, i\mathbf{h} = -\mathbf{h}i = \epsilon + i\}.$$

This number system generalizes the complex numbers ($i^2 = -1$), hyperbolic numbers ($\mathbf{h}^2 = 1$), and dual numbers ($\epsilon^2 = 0$), where i is the complex unit, ϵ is the dual unit, and $\mathbf{h}$ is the hyperbolic unit. These units are collectively referred to as *hybrid units*. According to (Özdemir, 2018), two hybrid numbers are considered equal if all their components are individually equal. The sum of two hybrid numbers is obtained by summing their corresponding components. The addition operation in the hybrid number system is both commutative and associative. The zero element acts as the additive identity. For any hybrid number k, its symmetric element under addition is $-k$, which is defined by negating all of its components. This shows that $(\mathbb{K}, +)$ forms an Abelian group. The conjugate of a hybrid number $k = a + bi + c\epsilon + d\mathbf{h}$ is defined as

$$\overline{k} = a - bi - c\epsilon - d\mathbf{h}.$$

In recent years, this number system has gained traction in various fields of applied science, and several researchers have applied it in different contexts. For further applications of hybrid numbers, we refer the reader to (Özdemir, 2018; Öztürk; Özdemir, 2020; Akbıyık *et al.*, 2021).

More recently, Olariu in (Olariu, 2002) and (Olariu, 2000) introduced a concept of Tricomplex numbers which are expressed in the form $(x, y, z) = x + y\mathbf{i} + z\mathbf{j}$, where x, y, and z are real numbers and $\mathbf{i}$ and $\mathbf{j}$ are imaginary units. In the context of mathematics, a Tricomplex number represents an element of a number system that extends the complex numbers. In a manner comparable to complex numbers, which have a real part and an imaginary part, Tricomplex numbers are defined by two imaginary parts in addition to a real part. First, consider $\mathbb{T}$ the ring of real Tricomplex numbers, that is, the set of ordered triples of real numbers (x, y, z), with the operations of addition $(+)$, and multiplication $(\times)$ given by:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \ , \tag{1}$$

and

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (x_1 x_2 + y_1 z_2 + z_1 y_2, z_1 z_2 + x_1 y_2 + y_1 x_2, y_1 y_2 + x_1 z_2 + z_1 x_2) \ ; \tag{2}$$

where $(x_i, y_i, z_i) \in \mathbb{R}^3$ for all $i = 1, 2$ and 3. According to Equation (1), the sum of the Tricomplex numbers (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the Tricomplex number $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$. Similarly, according to Equation (2), the product of the Tricomplex numbers (x_1, y_1, z_1) and (x_2, y_2, z_2) is the Tricomplex number $(x_1 x_2 + y_1 z_2 + z_1 y_2, z_1 z_2 + x_1 y_2 + y_1 x_2, y_1 y_2 + x_1 z_2 + z_1 x_2)$. Note that the multiplication rules for the imaginary units is given by $\mathbf{ij} = \mathbf{ji} = 1$, $\mathbf{i}^2 = \mathbf{j}$, and $\mathbf{j}^2 = \mathbf{i}$.

The Tricomplex numbers and their operations can be represented as $\mathbb{T} = (\mathbb{T}, +, \times)$. It is straightforward to verify, through direct calculation, that the Tricomplex zero is represented by the vector $(0, 0, 0)$, denoted by the symbol $\mathbf{0}$. Similarly, the Tricomplex unity is represented by the point $(1, 0, 0)$, denoted by the symbol $\mathbf{1}$. Furthermore, it can be demonstrated that the Tricomplex ring is both commutative and a unit ring (see (Olariu, 2000), (Mondal; Pramanik, 2015), and (Ottoni; Deus; Ottoni, 2024), along with their references). Therefore, the Tricomplex ring $\mathbb{T}$ can be described as a "symmetric" ring with respect to the ring of integers, embedded within the three-dimensional space $\mathbb{R}^3$. By "symmetric", we mean that the basic laws of arithmetic hold in $\mathbb{T}$, making it a natural domain for defining numerical sequences. For numerical sets that have a similar algebraic structure to the integers, see (Alves; Oliveira; Strey, 2023; Felzenszwalb, 1979; Hefez; Villela, 2012; Moura; Oliveira; Strey, 2022; Santos, 1998), and closely related references therein.

Tricomplex numbers are a powerful mathematical construct, particularly useful in the study of three-dimensional systems. They also have potential applications in physics and engineering, where three-dimensional models are a fundamental aspect of many phenomena. Additionally, in (Richter, 2022), the work explores a four-dimensional complex algebraic structure, which has applications in constructing directional probability distributions. In summary, Tricomplex numbers provide a valuable framework for understanding and modeling three-dimensional systems, with potential applications extending beyond mathematics into physics and engineering.

3. COMPLEX-TRICOMPLEX RING

In this text we assume that the reader has minimal knowledge of the algebraic structure of complexes, so we assume that $(\mathbb{C}, +, \cdot)$ is a field. For readers less experienced in abstract mathematics, we recommend (Beites, 2018; Domingues; Iezzi, 2003; Felzenszwalb, 1979; Hefez, 1993; Lang, 2005; Lequain; Garcia, 1983; Lubeck, 2024) for looking up some definitions or concepts.

In this section we denote by $\mathbb{TQ}$ the set with the vector $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$, that is, each component α_i, $i = 1, 2$ and 3, are complex elements, with the ordinary operations of addition $(+)$:

$$(\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2)\,, \tag{3}$$

and multiplication $(\times)$ given by:

$$(\alpha_1, \beta_1, \gamma_1) \times (\alpha_2, \beta_2, \gamma_2) = (\alpha_1\alpha_2 + \beta_1\gamma_2 + \gamma_1\beta_2, \gamma_1\gamma_2 + \alpha_1\beta_2 + \beta_1\alpha_2, \beta_1\beta_2 + \alpha_1\gamma_2 + \gamma_1\alpha_2)\,, \tag{4}$$

where $(\alpha_i, \beta_i, \gamma_i) \in \mathbb{C}^3$ for all $i = 1, 2$ and 3.

Example 1. *For instance, for vectors* $(22 + 8i,\ 3 + 25i,\ 31 + 3i)$ *and* $(4 - 25i,\ 72 + 5i,\ 4 - 52i)$ *in* $\mathbb{TQ}$*, we get:*

$$\begin{aligned} & (22 + 8i,\ 3 + 25i,\ 31 + 3i) + (4 - 25i,\ 72 + 5i,\ 4 - 52i) \\ = \ & ([22 + 4] = [8 - 25]i,\ [3 + 72] + [25 + 5]i,\ [31 + 4] + [3 - 52]i) \\ = \ & (26 - 17i,\ 75 + 30i,\ 35 - 49i)\,, \end{aligned}$$

and

$$
\begin{aligned}
& (22+8i,\ 3+25i,\ 31+3i)\times(4-25i,\ 72+5i,\ 4-52i) \\
= & ([22+8i][4-25i]+[3+25i][4-52i]+[31+3i][72+5i], \\
& [31+3i][4-52i]+[22+8i][72+5i]+[3+25i][4-25i], \\
& [3+25i][72+5i]+[22+8i][4-52i]+[31+3i][4-25i]) \\
= & ([22.4+8.25]+[8.4-22.25]i+[3.4+25.52]+[25.4-3.52]i \\
& +[31.72-3.5]+[31.5+3.72]i, [31.4+3.52]+[3.4-31.52]i \\
+ & [22.72-8.5]+[22.5+8.72]i+[3.4+25.25]+[25.4-3.25]i, \\
& [3.72-25.5]+[3.5+25.72]i+[22.4+8.52]+[8.4-22.52]i \\
& +[31.4+3.25]+[3.4-31.25]i) \\
= & (288-518i+1312-56i+2217+371i, 280-1600i+1544+686i+637+25i, \\
& 91+1815i+504-1112i+199-763i) \\
= & (3817-203i, 2461-889i, 794-60i).
\end{aligned}
$$

The aim of this paper is to ascertain whether the proposed extension of the complex numbers, designated as $\mathbb{TQ}$, with the operations defined in Equations (3) and (4) satisfies the fundamental laws of arithmetic, that is:

Theorem 2. *Let $\mathbb{TQ}$ be the set of $(\alpha, \beta, \gamma) \in \mathbb{C}^3$. Then $(\mathbb{TQ}, +, \times)$ is a commutative ring with unit, where the addition and multiplication operations are defined respectively in the equations (3) and (4) .*

The Theorem 2 is a direct consequence of the following results.

Proposition 3. *Let $\mathbb{TQ}$ be the set of $(\alpha, \beta, \gamma) \in \mathbb{C}^3$. Then $(\mathbb{TQ}, +)$ is an abelian group, with the addition operation defined in the equations (3).*

Proof. Let $\mathbb{TQ}$ be the set of ordered triples of complex numbers equipped with the addition operation (3), that is, defined as the usual vector sum in $\mathbb{C}^3$. So, given $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ in $\mathbb{TQ}$, the sum is defined as the ternary sum $(\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2)$. It is easy to see that this ordinary addition satisfies the following properties, as long as this property applies to each component of the complex field:

- (A1): Associativity of addition, that is,

$$[(\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2)] + (\alpha_3, \beta_3, \gamma_3) = (\alpha_1, \beta_1, \gamma_1) + [(\alpha_2, \beta_2, \gamma_2) + (\alpha_3, \beta_3, \gamma_3)] ;$$

- (A2): Commutativity of addition, that is,

$$(\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) = (\alpha_2, \beta_2, \gamma_2) + (\alpha_1, \beta_1, \gamma_1) ;$$

- (A3): Existence of a neutral element for addition, that is, there exists $(0, 0, 0) \in \mathbb{TQ}$, such that for any $(\alpha, \beta, \gamma) \in \mathbb{TQ}$,

$$(\alpha, \beta, \gamma) + (0, 0, 0) = (\alpha, \beta, \gamma) ;$$

- (A4): Existence of the opposite element, that is, for every $(\alpha, \beta, \gamma) \in \mathbb{TQ}$, there exists the element $(-\alpha, -\beta, -\gamma) \in \mathbb{TQ}$, called the opposite element of $(\alpha, \beta, \gamma) \in \mathbb{TQ}$, such that

$$(\alpha, \beta, \gamma) + (-\alpha, -\beta, -\gamma) = (0, 0, 0) .$$

From the properties mentioned above, and according to the Definition in (Domingues; Iezzi, 2003; Hefez; Villela, 2012) , $(\mathbb{TQ}, +)$ is an abelian group. □

In the next result, we will check the associativity property of multiplication in $\mathbb{TQ}$.

Proposition 4. *For all* $A = (\alpha_1, \beta_1, \gamma_1)$, $B = (\alpha_2, \beta_2, \gamma_2)$ *and* $C = (\alpha_3, \beta_3, \gamma_3)$ *in* $\mathbb{TQ}$, *the identity* $(A \times B) \times C = A \times (B \times C)$ *holds.*

Proof. A direct calculation shows that:

$$
\begin{aligned}
& (A \times B) \times C \\
= \; & [(\alpha_1, \beta_1, \gamma_1) \times (\alpha_2, \beta_2, \gamma_2)] \times (\alpha_3, \beta_3, \gamma_3) \\
= \; & [(\alpha_1\alpha_2 + \beta_1\gamma_2 + \gamma_1\beta_2, \gamma_1\gamma_2 + \alpha_1\beta_2 + \beta_1\alpha_2, \beta_1\beta_2 + \alpha_1\gamma_2 + \gamma_1\alpha_2)] \times (\alpha_3, \beta_3, \gamma_3) \\
= \; & \big((\alpha_1\alpha_2 + \beta_1\gamma_2 + \gamma_1\beta_2)\alpha_3 + (\gamma_1\gamma_2 + \alpha_1\beta_2 + \beta_1\alpha_2)\gamma_3 + (\beta_1\beta_2 + \alpha_1\gamma_2 + \gamma_1\alpha_2)\beta_3, \\
& (\beta_1\beta_2 + \alpha_1\gamma_2 + \gamma_1\alpha_2)\gamma_3 + (\alpha_1\alpha_2 + \beta_1\gamma_2 + \gamma_1\beta_2)\beta_3 + (\gamma_1\gamma_2 + \alpha_1\beta_2 + \beta_1\alpha_2)\alpha_3, \\
& (\gamma_1\gamma_2 + \alpha_1\beta_2 + \beta_1\alpha_2)\beta_3 + (\alpha_1\alpha_2 + \beta_1\gamma_2 + \gamma_1\beta_2)\gamma_3 + (\beta_1\beta_2 + \alpha_1\gamma_2 + \gamma_1\alpha_2)\alpha_3\big) \\
= \; & \big(\alpha_1\alpha_2\alpha_3 + \alpha_1\beta_2\gamma_3 + \alpha_1\gamma_2\beta_3 + \beta_1\beta_2\beta_3 + \gamma_1\beta_2\alpha_3 + \gamma_1\gamma_2\gamma_3 + \beta_1\alpha_2\gamma_3 + \gamma_1\alpha_2\beta_3, \\
& \beta_1\beta_2\gamma_3 + \beta_1\gamma_2\gamma_3 + \beta_1\alpha_2\alpha_3 + \alpha_1\gamma_2\gamma_3 + \gamma_1\alpha_2\gamma_3 + \alpha_1\alpha_2\beta_3 + \gamma_1\beta_2\beta_3 + \gamma_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3, \\
& \gamma_3\alpha_1\alpha_2 + \beta_1\gamma_2\gamma_3 + \gamma_1\beta_2\gamma_3 + \gamma_1\gamma_2\beta_3 + \alpha_1\beta_2\beta_3 + \beta_1\alpha_2\beta_3 + \beta_1\beta_2\alpha_3 + \alpha_1\gamma_2\alpha_3 + \gamma_1\alpha_2\alpha_3\big) \\
= \; & \big(\alpha_1(\alpha_2\alpha_3 + \beta_2\gamma_3 + \gamma_2\beta_3) + \beta_1(\gamma_2\alpha_3 + \alpha_2\gamma_3 + \beta_2\beta_3) + \gamma_1(\beta_2\alpha_3 + \gamma_2\gamma_3 + \alpha_2\beta_3), \\
& \gamma_1(\alpha_2\gamma_3 + \beta_2\beta_3 + \alpha_2\alpha_3) + \alpha_1(\gamma_2\gamma_3 + \alpha_2\beta_3 + \beta_2\alpha_3) + \beta_1(\beta_2\gamma_3 + \gamma_2\beta_3 + \alpha_2\alpha_3), \\
& \beta_1(\gamma_2\gamma_3 + \alpha_2\beta_3 + \beta_2\alpha_3) + \alpha_1(\gamma_3\alpha_2 + \beta_2\beta_3 + \gamma_2\alpha_3) + \gamma_1(\beta_2\gamma_3 + \gamma_2\beta_3 + \alpha_2\beta_3)\big) \\
= \; & (\alpha_1, \beta_1, \gamma_1) \times (\alpha_2\alpha_3 + \beta_2\gamma_3 + \gamma_2\beta_3, \gamma_2\gamma_3 + \alpha_2\beta_3 + \beta_2\alpha_3, \beta_2\beta_3 + \alpha_2\gamma_3 + \gamma_2\alpha_3) \\
= \; & (\alpha_1, \beta_1, \gamma_1) \times [(\alpha_2, \beta_2, \gamma_2) \times (\alpha_3, \beta_3, \gamma_3)] \\
= \; & A \times (B \times C)\,,
\end{aligned}
$$

this completes the proof. □

The following result examines the distributive property of multiplication in relation to addition in the context of the mathematical construction of the ring $\mathbb{TQ}$.

Proposition 5. *For all* $A = (\alpha_1, \beta_1, \gamma_1)$, $B = (\alpha_2, \beta_2, \gamma_2)$ *and* $C = (\alpha_3, \beta_3, \gamma_3)$ *in* $\mathbb{TQ}$, *the identity* $A \times (B + C) = (A \times B) + (A \times C)$ *holds.*

Proof. Note that

$$
\begin{aligned}
& A \times (B + C) \\
= \; & (\alpha_1, \beta_1, \gamma_1) \times [(\alpha_2, \beta_2, \gamma_2) + (\alpha_3, \beta_3, \gamma_3)] \\
= \; & (\alpha_1, \beta_1, \gamma_1) \times (\alpha_2 + \alpha_3, \beta_2 + \beta_3, \gamma_2 + \gamma_3) \\
= \; & (\alpha_1(\alpha_2 + \alpha_3) + \beta_1(\gamma_2 + \gamma_3) + \gamma_1(\beta_2 + \beta_3), \\
& \gamma_1(\gamma_2 + \gamma_3) + \alpha_1(\beta_2 + \beta_3) + \beta_1(\alpha_2 + \alpha_3), \\
& \beta_1(\beta_2 + \beta_3) + \alpha_1(\gamma_2 + \gamma_3) + \gamma_1(\alpha_2 + \alpha_3)) \\
= \; & (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \beta_1\gamma_2 + \beta_1\gamma_3 + \gamma_1\beta_2 + \gamma_1\beta_3, \\
& \gamma_1\gamma_2 + \gamma_1\gamma_3 + \alpha_1\beta_2 + \alpha_1\beta_3 + \beta_1\alpha_2 + \beta_1\alpha_3, \\
& \beta_1\beta_2 + \beta_1\beta_3 + \alpha_1\gamma_2 + \alpha_1\gamma_3 + \gamma_1\alpha_2 + \gamma_1\alpha_3) \\
= \; & (\alpha_1\alpha_2 + \beta_1\gamma_2 + \gamma_1\beta_2, \gamma_1\gamma_2 + \alpha_1\beta_2 + \beta_1\alpha_2, \beta_1\beta_2 + \alpha_1\gamma_2 + \gamma_1\alpha_2) + \\
& (\alpha_1\alpha_3 + \beta_1\gamma_3 + \gamma_1\beta_3, \gamma_1\gamma_3 + \alpha_1\beta_3 + \beta_1\alpha_3, \beta_1\beta_3 + \alpha_1\gamma_3 + \gamma_1\alpha_3) \\
= \; & (\alpha_1, \beta_1, \gamma_1) \times (\alpha_2, \beta_2, \gamma_2) + (\alpha_1, \beta_1, \gamma_1)(\alpha_3, \beta_3, \gamma_3) \\
= \; & (A \times B) + (A \times C)\,,
\end{aligned}
$$

which verifies the result. □

Combining the Propositions 3, 4 and 5, we conclude that the set of Tricomplex numbers with complex entries $\mathbb{TQ}$ and with the operations addition and multiplication, respectively by Equations (3) and (4), forms a ring, that is, the *Complex-Tricomplex* $(\mathbb{TQ}, +, \times)$ it is a ring.

Now, the commutativity property of multiplication in the context of the ring $\mathbb{TQ}$ will now be examined.

Proposition 6. *The multiplication in $\mathbb{TQ}$ is commutative, that is, $A \times B = B \times A$, with $A = (\alpha_1, \beta_1, \gamma_1)$ and $B = (\alpha_2, \beta_2, \gamma_2)$ in $\mathbb{TQ}$.*

Proof. A straightforward calculation demonstrates that:

$$\begin{aligned}
A \times B &= (\alpha_1, \beta_1, \gamma_1) \times (\alpha_2, \beta_2, \gamma_2) \\
&= (\alpha_1\alpha_2 + \beta_1\gamma_2 + \gamma_1\beta_2, \gamma_1\gamma_2 + \alpha_1\beta_2 + \beta_1\alpha_2, \beta_1\beta_2 + \alpha_1\gamma_2 + \gamma_1\alpha_2) \\
&= (\alpha_2\alpha_1 + \beta_2\gamma_1 + \gamma_2\beta_1, \gamma_2\gamma_1 + \alpha_2\beta_1 + \beta_2\alpha_1, \beta_2\beta_1 + \alpha_2\gamma_1 + \gamma_2\alpha_1) \\
&= (\alpha_2, \beta_2, \gamma_2) \times (\alpha_1, \beta_1, \gamma_1) \\
&= B \times A\,,
\end{aligned}$$

as required. □

For illustrative purposes, consider the example.

Example 7. *Consider the vectors $A = (22 + 8i,\ 3 + 25i,\ 31 + 3i)$ and $B = (4 - 25i,\ 72 + 5i,\ 4 - 52i)$ in $\mathbb{TQ}$. According to Example 1, we get*

$$\begin{aligned}
A \times B &= (22 + 8i, 3 + 25i, 31 + 3i) \times (4 - 25i, 72 + 5i, 4 - 52i) \\
&= (3817 - 203i, 2461 - 889i, 794 - 60i)\,. \qquad (5)
\end{aligned}$$

Also note that,

$$\begin{aligned}
& B \times A \\
=\ & (4 - 25i,\ 72 + 5i,\ 4 - 52i) \times (22 + 8i,\ 3 + 25i,\ 31 + 3i) \\
=\ & ([4 - 25i][22 + 8i] + [72 + 5i][31 + 3i] + [4 - 52i][3 + 25i], \\
& [4 - 52i][31 + 3i] + [4 - 25i][3 + 25i] + [72 + 5i][22 + 8i], \\
& [72 + 5i][3 + 25i] + [4 - 25i][31 + 3i] + [4 - 52i][22 + 8i]) \\
=\ & ([4.22 + 25.8] + [4.8 - 25.22]i + [72.31 - 5.3] + [72.3 + 5.31]i \\
& +[4.3 + 52.25] + [4.25 - 52.3]i, [4.31 + 52.3] + [4.3 - 52.31]i \\
& +[4.3 + 25.25] + [4.25 - 25.3]i + [72.22 - 5.8] + [72.8 + 5.22]i, \\
& [72.3 - 5.25] + [72.25 + 5.3]i + [4.31 + 25.3] + [4.3 - 25.31]i \\
& +[4.22 + 52.8] + [4.8 - 52.22]i) \\
=\ & (288 - 518i + 2217 + 371i + 1312 - 56i,\ 280 - 1600i + 637 + 25i + 1544 + 686i, \\
& 91 + 1815i + 199 - 763i + 504 - 1112i) \\
=\ & (3817 - 203i,\ 2461 - 889i,\ 794 - 60i), \qquad (6)
\end{aligned}$$

this result (6) matches the value obtained for (5).

Finally, we prove that $\mathbb{TQ}$ is a unity ring, that is, the element $(1_{\mathbb{C}}, 0, 0)$ in $\mathbb{TQ}$ is the unity.

Proposition 8. *For every element $(\alpha, \beta, \gamma) \in \mathbb{TQ}$, the identity $(\alpha, \beta, \gamma) \times (1_{\mathbb{C}}, 0, 0) = (\alpha, \beta, \gamma)$ holds.*

Proof. See that:

$$\begin{aligned}(\alpha,\beta,\gamma)\times(1_{\mathbb{C}},0,0) &= (\alpha,\beta,\gamma)\times(1_{\mathbb{C}},0,0)\\ &= (\alpha 1_{\mathbb{C}}+\beta 0+\gamma 0,\gamma 0+\alpha 0+\beta 1_{\mathbb{C}},\beta 0+\alpha 0+\gamma 1_{\mathbb{C}})\\ &= (\alpha+0+0,0+\beta+0,0+0+1_{\mathbb{C}}\alpha)\\ &= (\alpha,\beta,\gamma)\,,\end{aligned}$$

as needed. □

Remark 9. *It should be noted that the ring Complex-Tricomplex, denoted by* $\mathbb{TQ}$, *is not an integrity ring. To illustrate this, consider the following example, consider* $A=(1_{\mathbb{C}},-1_{\mathbb{C}},0)$ *and* $B=(1_{\mathbb{C}},1_{\mathbb{C}},1_{\mathbb{C}})$. *According to Equation (2), the product* $\alpha\times\beta$ *is*

$$\begin{aligned}A\times B &= (1_{\mathbb{C}},-1_{\mathbb{C}},0)\times(1_{\mathbb{C}},1_{\mathbb{C}},1_{\mathbb{C}})\\ &= (1_{\mathbb{C}}-1_{\mathbb{C}}+0,0-1_{\mathbb{C}}+1_{\mathbb{C}},-1_{\mathbb{C}}+1+0)\\ &= (0+0+0,0+0+0,0+0+0)\\ &= (0,0,0)\,,\end{aligned}$$

which completes our argument, since we have two non-null elements in $\mathbb{TQ}$ *whose product is null.*

4. VECTOR SPACES OVER THE FIELD OF COMPLEX NUMBERS

Moreover, we can conclude that $\mathbb{TQ}$ is a vector space over the field of complex numbers $\mathbb{C}$. This means it is a mathematical structure consisting of the set $\mathbb{TQ}$, along with two operations: *vector addition*, as defined in Equation (1), and *scalar multiplication*, as defined by

$$\alpha\cdot(\alpha_1,\beta_1,\gamma_1)=(\alpha\cdot\alpha_1,\alpha\cdot\beta_1,\alpha\cdot\gamma_1)$$

for all $\alpha\in\mathbb{C}$ and $A=(\alpha_1,\beta_1,\gamma_1)\in\mathbb{TQ}$. This structure $(\mathbb{TQ},+,\cdot)$ satisfies the following axioms of a vector space. For $A,B,C\in\mathbb{TQ}$ and $\alpha,\beta\in\mathbb{C}$, the following properties are verified:

- *Associativity of addition:* $(A+B)+C=A+(B+C)$.
- *Additive identity:* There exists $0\in\mathbb{TQ}$ such that $B+0=B$ for all $B\in\mathbb{TQ}$.
- *Additive inverse:* For every $B\in\mathbb{TQ}$, there exists $-B\in\mathbb{TQ}$ such that $B+(-B)=0$.
- *Commutativity of addition:* $A+B=B+A$.
- *Associativity of scalar multiplication:* $\alpha\cdot(\beta\cdot B)=(\alpha\cdot\beta)\cdot B$.
- *Multiplicative identity:* $1_{\mathbb{C}}\cdot B=B$, where $1_{\mathbb{C}}$ is the identity element of $\mathbb{C}$.
- *Distributivity of scalar over vectors:* $\alpha\cdot(A+B)=\alpha\cdot A+\alpha\cdot B$.
- *Distributivity of scalar over scalars:* $(\alpha+\beta)\cdot B=\alpha\cdot B+\beta\cdot B$.

As a vector space, $(\mathbb{TQ},+,\cdot)$ has dimension 3 over the field of complex numbers $\mathbb{C}$, since any vector $(\alpha,\beta,\gamma)\in\mathbb{TQ}$ can be expressed as a linear combination of the three linearly independent canonical vectors $(1_{\mathbb{C}},0,0)$, $(0,1_{\mathbb{C}},0)$, and $(0,0,1_{\mathbb{C}})$. Specifically, we have:

$$(\alpha,\beta,\gamma)=\alpha(1_{\mathbb{C}},0,0)+\beta(0,1_{\mathbb{C}},0)+\gamma(0,0,1_{\mathbb{C}})\,.$$

5. FINAL CONSIDERATIONS

This work introduced an extension of the real Tricomplex ring, as defined by Olariu(Olariu, 2002; Olariu, 2000), representing a three-dimensional number system over the real numbers and extending the concept of complex numbers. Based on this foundation, a Complex-Tricomplex ring was defined as a three-dimensional number system over the field of complex numbers. It was demonstrated that the $\mathbb{TQ}$ ring shares an algebraic structure analogous to that of the ring of integers. According (Mondal; Pramanik, 2015; Richter, 2022) the Tricomplex ring $\mathbb{T}$ has numerous applications, including its role in verifying a sequence in three-dimensional space that reflects the properties of the Fibonacci sequence, known as the Fibonacci Tricomplex sequence, see (Costa *et al.*, 2024); and in (Costa; Catarino; Santos, 2025) the authors examine the symmetrical properties for the Tricomplex Repunit sequence to those we know for the ordinary repunit sequence. Similarly, the ring $\mathbb{TQ}$ can serve as a support ring for identifying or defining new (extensions or generalizations) Complex-Tricomplex sequences. It is hoped that this system will continue to be explored by researchers in applied sciences. In future works, we believe that this extension of the Tricomplex ring $\mathbb{TC}$ could serve as a foundation for defining extensions of numerical sequences. Furthermore, it aims to enable the extension or generalization of certain sequences to these sets of numbers.

ACKNOWLEDGMENTS

The first author expresses her sincere thanks to the Federal University of Tocantins (Campus Arraias and PROPESQ) for its valuable support.

The authors would like to thank the referees for their helpful suggestions and comments.

6. REFERÊNCIAS

AKBIYIK, M.; AKBIYIK, S. Y.; KARACA, E.; YILMAZ, F. De moivre's and euler formulas for matrices of hybrid numbers. **Axioms**, MDPI, v. 10, n. 3, p. 213, 2021.

ALVES, L. C.; OLIVEIRA, E. J. T. de; STREY, E. Inteiros gaussianos. **Professor de Matemática Online**, v. 11, n. 3, p. 361–379, 2023.

BAUMGART, J. K. **Tópicos de história da matemática para uso em sala de aula: álgebra**. São Paulo: Atual Editora, 1992.

BEITES, P. (não) são números complexos. **Gazeta de Matemática**, SPM, n. 185, p. 28–32, 2018.

BOYER, C. B.; MERZBACH, U. C. **A history of mathematics**. New York: John Wiley and Sons, 2011.

COSTA, E. A.; BASTOS, R. Colocando ordem nos complexos. **Colloquium Exactarum**, v. 4, n. 1, p. 33–38, 2012.

COSTA, E. A.; CATARINO, P. M.; MONTEIRO, F. S.; SOUZA, V. M.; SANTOS, D. C. Tricomplex fibonacci numbers: A new family of fibonacci-type sequences. **Mathematics**, MDPI, v. 12, n. 23, p. 3723, 2024.

COSTA, E. A.; CATARINO, P. M.; SANTOS, D. C. A study of the symmetry of the tricomplex repunit sequence with repunit sequence. **Symmetry**, MDPI, v. 17, n. 1, p. 28, 2025.

DOMINGUES, H. H.; IEZZI, G. **Álgebra moderna**. São Paulo: Atual, 2003.

EVES, H. **Introdução à história da matemática**. Campinas: Editora da UNICAMP, 2008.

FELZENSZWALB, B. **Álgebras de dimensão finita**. [S.l.]: Instituto de Matemática Pura e Aplicada (Colóquio Brasileiro de Matemática), 1979.

HEFEZ, A. **Curso de Algebra**. Rio de Janeiro: Sociedade Brasileira de Matemática, 1993. (Coleçao Matemática Universitária, IMPA, CNPq, 1).

HEFEZ, A.; VILLELA, M. L. T. **Polinômios e equações algébricas**. Rio de Janeiro: Sociedade Brasileira de Matemática, 2012. (Coleção ProfMat).

LANG, S. **Undergraduate algebra**. New York: Springer Science & Business Media, 2005.

LEQUAIN, Y.; GARCIA, A. **Álgebra: uma introdução**. Rio de Janeiro: Instituto de Matemática Pura e Aplicada, 1983. (Matemática universitária, 39).

LUBECK, K. R. M. Roger cotes e sua versão da fórmula de euler. **Revista de Matemática da UFOP**, v. 1, 2024.

MILIES, C. P. **Breve história da álgebra abstrata**. Salvador, Universidade Federal da Bahia: Minicurso apresentado na II Bienal da Sociedade Brasileira de Matemática–SBM, 2004.

MONDAL, K.; PRAMANIK, S. Tri-complex rough neutrosophic similarity measure and its application in multi-attribute decision making. **Critical Review**, v. 11, p. 26–40, 2015.

MONTEIRO, L. H. J. **Elementos de álgebra**. Rio de Janeiro: IMPA-Livro Técnico, 1969.

MOURA, R. P.; OLIVEIRA, E. J. T. de; STREY, E. Inteiros de eisenstein. **Professor de Matemática Online**, v. 10, n. 3, p. 420–439, 2022.

OLARIU, S. Complex numbers in three dimensions. **arXiv**, 2000. Preprint math/0008120. Disponível em: <https://doi.org/10.48550/arXiv.math/0008120>.

OLARIU, S. **Complex numbers in n dimensions**. Amsterdam, Netherlands: Elsevier Science B.V., 2002. (Mathematics Studies).

OTTONI, A.; DEUS, N. C. L.; OTTONI, J. E. O. A álgebra dos números ternários. **Revista de Matemática da UFOP**, v. 1, 2024.

ÖZDEMIR, M. Introduction to hybrid numbers. **Advances in Applied Clifford Algebras**, v. 28, p. 1–32, 2018.

ÖZTÜRK, İ.; ÖZDEMIR, M. Similarity of hybrid numbers. **Mathematical Methods in the Applied Sciences**, Wiley Online Library, v. 43, n. 15, p. 8867–8881, 2020.

RICHTER, W. On complex numbers in higher dimensions. **Axioms**, MDPI, v. 22, n. 11.1, 2022.

ROQUE, T. **História da matemática**. Rio de Janeiro: Zahar, 2012.

ROQUE, T.; CARVALHO, J. B. P. **Tópicos de história da matemática**. Rio de Janeiro: Sociedade Brasileira de Matemática, 2012.

SANTOS, J. C. Sobre a realidade dos números complexos. **Gazeta de Matemática**, SPM, v. 200, p. 25–27, 2023.

SANTOS, J. P. O. **Introdução à teoria dos números**. Rio de Janeiro: IMPA, 1998. v. 128.