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Um Teorema de Extensão CR para uma Classe de Hipersuperfícies Reais

A CR Extension Theorem for a Class of Real Hypersurfaces

Un Teorema de Extensión CR para una Clase de Hipersuperficies Reales

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Resumo

Sabe-se que nem toda função CR admite extensão holomorfa. Este trabalho propõe um novo teorema de extensão CR para a hipersuperfície $\{(z,w) \in \mathbb{C}^2; \text{Im } z = \phi(w,\overline{w})\}$, com condições sobre a função polinomial $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$. A demonstração deste fato está calcada no Teorema de Extensão de Hans Lewys assim como nas propriedades da forma de Levi da hipersuperfície. Como resultado principal, mostra-se que a hipersuperfície admite extensão CR sempre que as derivadas segundas mistas são não nulas, estabelecendo uma caracterização geométrica das condições necessárias à extensão holomorfa.

Palavras-chave: *Hipersuperfície. Extensão. Funções CR. Forma de Levi.*

Abstract

It is known that not every CR function admits a holomorphic extension. This paper proposes a new CR extension theorem for the hypersurface $\{(z,w) \in \mathbb{C}^2; \text{Im } z = \phi(w,\overline{w})\}$, with conditions on the polynomial function $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$. The proof of this fact is based on Hans Lewy's Extension Theorem as well as on the properties of the Levi form of the hypersurface. As main result, it is shown that the hypersurface admits a CR extension whenever the mixed second derivatives are nonzero, thereby establishing a geometric characterization of the necessary conditions for holomorphic extension.

Keywords: *Hypersurface. Extension. CR functions. Levi form.*

Resumen

Se sabe que no toda función CR admite una extensión holomorfa. Este trabajo propone un nuevo teorema de extensión CR para la hipersuperficie $\{(z,w) \in \mathbb{C}^2; \text{Im } z = \phi(w,\overline{w})\}$, con condiciones sobre la función polinomial $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$. La demostración de este hecho se basa en el Teorema de Extensión de Hans Lewy, así como en las propiedades de la forma de Levi de la hipersuperficie. Como resultado principal, se demuestra que la hipersuperficie admite una extensión CR siempre que las derivadas segundas mixtas sean no nulas, estableciendo así una caracterización geométrica de las condiciones necesarias para la extensión holomorfa.

Palabras-Clave: *Hipersuperficie. Extensión. Funciones CR. Forma de Levi.*

1. INTRODUCTION

The study of CR structures and their associated submanifolds has been a cornerstone in complex analysis and differential geometry. CR functions, defined on CR manifolds, naturally arise when considering the trace of holomorphic functions.

The classical problem about CR functions is to determine sufficient conditions that guarantee the existence of a holomorphic extension across the submanifold. In this scenario, understanding the geometry of the space that permit such extensions has been a subject of great interest since the foundational work of Hans Lewy in the mid-20th century (Lewy, 1956) and (Boggess, 1991).

Lewy's Extension Theorem provided crucial insights into the conditions under which CR functions defined on real submanifolds in complex spaces can be analytically continued, setting the stage for decades of further research.

In this context, the study of holomorphic extension problems is not only of theoretical interest but also has applications in several complex variables, PDEs (Partial Differential Equations), and mathematical physics.

Understanding the precise geometric and analytic conditions under which a CR function can be extended holomorphically is essential for advancing the field. This is particularly relevant for CR submanifolds, where the intricate interplay between the complex and real structures leads to rich mathematical phenomena.

This paper aims to explore new conditions under which CR functions defined on certain CR submanifolds can be extended holomorphically. Building on previous works, such as the classic results of Henkin and Kohn (Henkin; Kohn, 1995), this study delves into specific geometric configurations that facilitate or obstruct the extension process. The goal is to contribute to the broader understanding of the analytic continuation of CR functions by providing novel criteria and examples that features this complex relationship.

To achieve this, the paper is structured as follows: after a brief review of fundamental concepts related to CR structures and holomorphic functions, we discuss recent advancements in the field, particularly focusing on geometric conditions that are both necessary and sufficient for holomorphic extension. We then introduce our main results, which include new theorems that addresses some of the unresolved issues in this area. Finally, we present applications of our findings to specific classes of CR submanifolds, illustrating the practical implications of our theoretical work.

The goal of this paper is to prove a holomorphic extension theorem of CR functions for a class of real hypersurfaces. We know that every holomorphic function in $\mathbb{C}^n$ restricts to a CR function on a CR submanifold M of $\mathbb{C}^n$. However, not all CR functions are the restrictions of holomorphic functions. We shall determine which geometric conditions on M guarantee that CR functions on M extend as holomorphic functions on some open set Ω in $\mathbb{C}^n$.

The main objective of this paper is to define which necessary conditions the following manifold

$$M = \left\{(z, w) \in \mathbb{C}^2; \text{Im}\, z = \phi(w, \overline{w})\right\},$$

has to satisfy the Hans Lewy's Extension Theorem.

The results presented below in this section are formally stated and proven in the following references: (Boggess, 1991) and (Hörmander, 1983). For further details, the reader is referred to these sources.

Definition 1. *For a submanifold M of $\mathbb{C}^n$, $T_p(M)$ is the real tangent space of M at a point $p \in M$. In general, $T_p(M)$ is not invariant under the complex structure map J for $T_p(\mathbb{C}^n)$.*

Definition 2. *For a point $p \in M$, where M is a submanifold of $\mathbb{C}^n$, the complex tangent space of M at p is the vector space*

$$H_p(M) = T_p(M) \cap J\left\{T_p(M)\right\}.$$

The space $H_p(M)$ is called the holomorphic tangent space.

Definition 3. *The complexification of $T_p(M)$ and $H_p(M)$ are denoted by $T_p(M) \otimes \mathbb{C}$ and $H_p(M) \otimes \mathbb{C}$, respectively. Note that $H_p(M) \otimes \mathbb{C}$ is the direct sum of $+i$ and $-i$ eigenspaces of J, where J denotes the complex structure map, which are denoted by $H_p^{1,0}(M)$ and $H_p^{0,1}(M)$, respectively. We have,*

$$\begin{aligned} H_p^{1,0}(M) &= T_p^{1,0}(\mathbb{C}^n) \cap \{T_p(M) \otimes \mathbb{C}\} \\ H_p^{0,1}(M) &= T_p^{0,1}(\mathbb{C}^n) \cap \{T_p(M) \otimes \mathbb{C}\} \\ H_p^{0,1}(M) &= \overline{H_p^{1,0}(M)}. \end{aligned}$$

Definition 4. *IF M is a CR submanifold of $\mathbb{C}^n$, then the dimensions of $H_p^{1,0}(M), H_p^{0,1}(M)$ and $H_p(M) \otimes \mathbb{C}$ are independent of the point $p \in M$. We define the following subsets of $T^{\mathbb{C}}(M)$:*

$$\begin{aligned} H^{\mathbb{C}} &= \bigcup_{p \in M} H_p(M) \otimes \mathbb{C} \\ H^{1,0}(M) &= \bigcup_{p \in M} H_p^{1,0}(M) \\ H^{0,1}(M) &= \bigcup_{p \in M} H_p^{0,1}(M). \end{aligned}$$

Definition 5. *A submanifold M of $\mathbb{C}^n$ is called an imbedded CR manifold or a CR submanifold of $\mathbb{C}^n$ if $\dim_{\mathbb{R}} H_p(M)$ is independent of p in M.*

Definition 6. *A CR submanifold M is called generic if $\dim_{\mathbb{R}} H_p(M)$ is minimal.*

Theorem 7. *Suppose $M = \left\{(x+iy, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d}; y = h(x,w)\right\}$, where $h : \mathbb{R}^d \times \mathbb{C}^{n-d} \longrightarrow \mathbb{R}^d$ is of class C^k with $k \geq 2$, $h(0) = 0$ and $Dh(0) = 0$. A basis for $H^{1,0}(M)$ near the origin is given by $L_1, \ldots, L_{n-d}$ with*

$$L_j = \frac{\partial}{\partial w_j} + 2i \sum_{l=1}^{d} \left(\sum_{k=1}^{d} \mu_{lk} \frac{\partial h_k}{\partial w_j} \frac{\partial}{\partial z_l} \right), \;\; 1 \leq j \leq n-d,$$

where μ_{lk} is the (l,k)-entry element of the $d \times d$ matrix

$$\left(I - i\frac{\partial h}{\partial x} \right)^{-1}.$$

A basis for $H^{0,1}(M)$ near the origin is given by $\overline{L}_1, \ldots, \overline{L}_{n-d}$.

Demonstração. See (Boggess, 1991). □

Definition 8. *A CR submanifold of the form $M = \{(x+iy, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d}; y = h(w)\}$, where $h : \mathbb{C}^{n-d} \longrightarrow \mathbb{R}^d$ is smooth with $h(0) = 0$ and $Dh(0) = 0$ is called rigid.*

Definition 9. *A submanifold $M \subset \mathbb{C}^n$ defined by*

$$M = \left\{(x+iy, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d}; y = q(w, \overline{w})\right\},$$

where $q : \mathbb{C}^{n-d} \times \mathbb{C}^{n-d} \longrightarrow C^d$ is a quadratic form is called a quadric submanifold of $\mathbb{C}^n$.

Remark 10. From Theorem 7, the generators for $H^{1,0}(M)$ are

$$L_j = \frac{\partial}{\partial w_j} + 2i \sum_{l=1}^{d} \frac{\partial q_l}{\partial w_j} \frac{\partial}{\partial z_l}, \quad 1 \leq j \leq n-d.$$

and the generator for $H^{0,1}(M)$ is $\overline{L}_j = \dfrac{\partial}{\partial \overline{w}_j} - 2i \displaystyle\sum_{l=1}^{d} \frac{\partial \overline{q}_l}{\partial \overline{w}_j} \frac{\partial}{\partial \overline{z}_l}$.

Definition 11. *Let M be a C^∞ manifold and suppose $\mathbb{L}$ is a subbundle of $T^{\mathbb{C}}(M)$. The pair $(M, \mathbb{L})$ is called CR manifold or CR structure if*

- *(a) $\mathbb{L} \cap \overline{\mathbb{L}}_p = \{0\}$ for each $p \in M$.*
- *(b) $\mathbb{L}$ is involutive, that is, $[L_1, L_2] \in \mathbb{L}$ whenever $L_1, L_2 \in \mathbb{L}$.*

Definition 12. *Suppose $(M, \mathbb{L})$ is a CR structure. A function $f : M \longrightarrow \mathbb{C}$ is called a CR function if $\overline{\partial}_M f = 0$ on M.*

Lemma 13. *Suppose $(M, \mathbb{L})$ is a CR structure. A C^1 function $f : M \longrightarrow \mathbb{C}$ is CR if, and only if, $\overline{L} f = 0$ on M for all $\overline{L} \in \mathbb{L}$*

Lemma 14. *Suppose $M = \{z \in \mathbb{C}^n; \rho_1(z) = \cdots = \rho_d(z) = 0\}$ is a CR generic submanifold of C^n. A C^1 function $f : M \longrightarrow \mathbb{C}$ is CR if, and only if, $\overline{\partial} \widetilde{f} \wedge \overline{\partial} \rho_1 \wedge \ldots \wedge \overline{\partial} \rho_d = 0$ on M, where $\widetilde{f} : \mathbb{C}^n \longrightarrow \mathbb{C}$ is any C^1 extension of f.*

If M is a CR submanifold of $\mathbb{C}^n$ then any holomorphic function on a neighborhood of M in $\mathbb{C}^n$ restricts to a CR function on M by Lemma 14. But the converse is not true; that is, CR functions do not always extend as holomorphic functions, as the following example illustrates this:

Example 15. Suppose $M = \{(z, w) \in \mathbb{C}^2; \text{Im } z = 0\}$. Here, $H^{0,1}(M) = \overline{\mathbb{L}}$ is spanned by the vector field $\dfrac{\partial}{\partial \overline{w}}$. A function $f : M \longrightarrow \mathbb{C}$ is CR if

$$\frac{\partial f}{\partial \overline{w}}(x, w) = 0.$$

A CR function on M is a function that is holomorphic in w with $x \in \mathbb{R}$ held fixed. There is no extra condition on the behavior of a CR function in the x-variable. Thus, any arbitrary function of x is automatically CR. Therefore, any non-analytic function of x is an example of a CR function that does not extend to a holomorphic function in a neighborhood of M in $\mathbb{C}^2$.

2. EXTENSION THEOREMS

The theorems and lemmas presented in this section are well established in the literature. For theoretical foundations and detailed proofs, the reader is referred to references (Boggess, 1991), (Lewy, 1956) and (Hörmander, 1983).

Hans Lewy's CR Extension Theorem for hypersurfaces is the main theoretical result underpinning the present work. Originally proved by Hans Lewy, as presented in (Lewy, 1956), this theorem plays a central role in the analysis carried out throughout this paper and is essential for the formulation and understanding of the subsequent results.

Theorem 16. *Suppose M is a real analytic generic CR submanifold of $\mathbb{C}^n$ with real dimension at least n. Suppose $f : M \longrightarrow \mathbb{C}$ is a real analytic CR function on M. Then there is a neighborhood U of M in $\mathbb{C}^n$ and a unique holomorphic function $F : U \longrightarrow \mathbb{C}$ with $F|_M = f$.*

The statement of the theorem shows that any real analytic CR function on a real analytic CR submanifold of $\mathbb{C}^n$ extends holomorphically to an open set in $\mathbb{C}^n$ that may depend on the CR function. If there are no further geometric conditions on the CR submanifold, then the CR extension to an open set where the function is independent is impossible, even when the CR submanifold is real analytic. Note that the statement is different from what we intend to show, because we are interested in the CR extension to an open set that is independent of the function. In order to state the desired theorem, we need to define some objects that ensure the geometric conditions of the theorem.

The property *ii)* of Definition 11 states that in an abstract CR structure $(M, \mathbb{L})$, the bundle $\mathbb{L}$ must be involutive. We know the subbundle $\mathbb{L} \oplus \overline{\mathbb{L}} \subset T^{\mathbb{C}}(M)$ is not necessarily involutive. The Levi form for M is defined so that it measures the degree to which $\mathbb{L} \oplus \overline{\mathbb{L}}$ fails to be involutive.

Definition 17. *For $p \in M$, let*

$$\pi_p : T_p(M) \otimes \mathbb{C} \longrightarrow \frac{T_p(M) \otimes \mathbb{C}}{\mathbb{L}_p \oplus \overline{\mathbb{L}}_p}$$

be the natural projection map.

Definition 18. *The Levi form at a point $p \in M$ is the map $\mathcal{L}_p : \mathbb{L}_p \longrightarrow \dfrac{T_p(M) \otimes \mathbb{C}}{\mathbb{L}_p \oplus \overline{\mathbb{L}}_p}$ defined by*

$$\mathcal{L}_p(L_p) = \frac{1}{2i}\pi_p \left\{ \left[\overline{L}, L\right] \right\} \text{ for } L_p \in \mathbb{L}_p,$$

where L is any vector field in $\mathbb{L}$ that equals L_p at p. Note that $[\cdot, \cdot]$ denotes the Lie bracket.

The Levi form is well defined. In order to prove it, we must show that definition is independent of the $\mathbb{L}$-vector field extension of the vector $L_p \in \mathbb{L}_p$.

Lemma 19. *Suppose L and G are two vector fields in $\mathbb{L}$ with $L_p = G_p$, then*

$$\pi_p \left[\overline{L}, L\right] = \pi_p \left[\overline{G}, G\right].$$

Definition 20. *A CR structure $(M, \mathbb{L})$ is called Levi flat if the Levi form of M vanishes at each point on M.*

Remark 21. Returning to Example 15, we can observe that M is Levi *flat*, and this fact will be important when we study whether or not extension occurs in this example.

In fact, we know that M has generators for $\mathbb{L} = H^{1,0}(M)$ and $\overline{\mathbb{L}} = H^{0,1}(M)$ given by $\dfrac{\partial}{\partial w}$ and $\dfrac{\partial}{\partial \overline{w}}$, respectively.

Hence, $\left[\dfrac{\partial}{\partial w}, \dfrac{\partial}{\partial \overline{w}}\right] = 0$. Also note that M is foliated by the complex manifolds

$$M_x = \{(x, w); w \in \mathbb{C}\} \text{ para } x \in \mathbb{R}.$$

The complexified tangent bundle of each M_x is given by $\mathbb{L} \oplus \overline{\mathbb{L}}$. In order to generalize this fact, we have the following result.

Theorem 22. *Suppose that $(M, \mathbb{L})$ is a Levi flat CR structure. Then M is locally foliated by complex manifolds whose complexified tangent bundle is given by $\mathbb{L} \oplus \overline{\mathbb{L}}$.*

Definition 23. *A real hypersurface M is called strictly pseudoconvex at a point $p \in M$ if the Levi form at p is either positive or negative definite, that is, if there exists a defining function ρ for M so that*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_j \partial \overline{\zeta}_k}(p) w_j \overline{w}_k > 0, \tag{1}$$

for all $W = \displaystyle\sum_{j=1}^{n} w_j \left(\frac{\partial}{\partial \zeta_j}\right) \in H_p^{1,0}(M)$.

Theorem 24. *Suppose that $M \subset \mathbb{C}^n$ is a smooth real hypersurface that is strictly pseudoconvex at a point $p \in M$. Then there is a biholomorphic map F defined on a neighborhood U of $p \in \mathbb{C}^n$ so that $F\{M \cap U\}$ is a strictly convex hypersurface in $F\{U\} \subset \mathbb{C}^n$.*

As we said, every holomorphic function on $\mathbb{C}^n$ restricts to a CR function on a CR submanifold M of $\mathbb{C}^n$. However, not all CR functions are the restriction of holomorphic functions. In this section, we examine geometric conditions on M that guarantee that CR functions on M extend as holomorphic functions on some open set Ω in $\mathbb{C}^n$.

We are especially concerned with the CR extension to an open set that is function independent. For that, we need to answer the following question: Given an open set ω in M, does there exist an open set Ω in $\mathbb{C}^n$ such that each CR function on ω extends to a holomorphic function on Ω?

This question is different than answered by Theorem 16, which shows that any real analytic CR function on a real analytic CR submanifold of $\mathbb{C}^n$ holomorphically extends to an open set in $\mathbb{C}^n$ that may depend on the CR function. If there are no more conditions on the CR submanifold, then the CR extension to an open set that is function independent is impossible.

To illustrate this idea, we turn back in the Example 15. Note that each real analytic function on an open set $\omega \subset M$ extends to a holomorphic function on an open set $\Omega \subset \mathbb{C}^2$ with $\Omega \cap M = \omega$. On the other hand,

$$\omega = \bigcap_{\epsilon > 0} \Omega_\epsilon,$$

where

$$\Omega_\epsilon = \left\{(z, w) \in \mathbb{C}^2; (\text{Re } z, w) \in \omega \text{ and } |\text{Im } z| < \epsilon\right\}.$$

If ω is convex, then each Ω_ϵ is convex and hence a domain of holomorphy. From the Theory of Several Complex Variables, a holomorphic function $f_\epsilon : \Omega_\epsilon \longrightarrow \mathbb{C}$ exists that cannot be analytically continued past any part of the boundary of Ω_ϵ. The restriction of f_ϵ to ω is an example of a real analytic CR function on ω that cannot be analytically continued past any part of the boundary of Ω_ϵ. Since $\epsilon > 0$ is arbitrary, we conclude that there does not exist a single open set $\Omega \subset \mathbb{C}^2$ to which all CR functions on ω holomorphically extend.

We know by Observation 21, M is Levi flat. A similar construction of the Ω_ϵ can be carried out for any Levi flat submanifold in $\mathbb{C}^n$. This fact indicates that Levi form has a fundamental role in the CR extension.

Definition 25. *Let $M = \{z \in \mathbb{C}^n; \rho = 0\}$ be a hypersurface in $\mathbb{C}^n$, where $\rho : \mathbb{C}^n \longrightarrow \mathbb{R}$ is smooth with $d\rho \neq 0$ on M. If ρ is scaled so that $|\bigtriangledown \rho(p)| = 1$, then the Levi form of M at p is the map*

$$W \longmapsto \left(-\sum_{j,k=1}^{n} \frac{\partial^2 \rho(p)}{\partial \zeta_j \partial \overline{\zeta}_k} w_j \overline{w}_k \right) \bigtriangledown \rho(p) \text{ for } W = \sum_{j=1}^{n} w_j \frac{\partial}{\partial \zeta_j} \in H_p^{1,0}(M).$$

Definition 26. *Let $\Omega^+ = \{z \in \mathbb{C}^n; \rho(z) > 0\}$ and $\Omega^- = \{z \in \mathbb{C}^n; \rho(z) < 0\}$.*

Theorem 27 (Hans Lewy's CR Extension Theorem for Hypersurfaces)**.** *Let M be a real hypersurface in $\mathbb{C}^n$, $n \geq 0$ of class C^k, $3 \leq k \leq \infty$, and let p be a point in M.*

a) If the Levi form on M at p has at least one positive eigenvalue then for each open set ω in M with $p \in \omega$, there is an open set U in $\mathbb{C}^n$ with $p \in U$ such that for each CR function f of class C^1 on ω, there is a unique function F which is holomorphic on $U \cap \Omega^+$ and continuous on $U \cap \overline{\Omega^+}$ such that $F|_{U \cap M} = f$.

b) If the Levi form on M at p has at least one positive eigenvalue then for each open set ω in M with $p \in \omega$, there is an open set U in $\mathbb{C}^n$ with $p \in U$ such that for each CR function f of class C^1 on ω, there is a unique function F which is holomorphic on $U \cap \Omega^-$ and continuous on $U \cap \overline{\Omega^-}$ such that $F|_{U \cap M} = f$.

c) If the Levi form of M at p has eigenvalues of opposite sign, then for each open set ω in M with $p \in \omega$, there is an open set U in $\mathbb{C}^n$ with $p \in U$ such that each CR function of class C^1 on ω is the restriction on $U \cap \omega$ of a unique holomorphic function defined on U.

The parts *a)* and *b)* describes one sided CR extension results. The part *c)* describes a two sided CR extension result. Note that the quantifiers are arranged so that the open set U depends only on ω and not on the CR function defined there. Moreover, since holomorphic functions are real analytic, the part *c)* implies the following regularity result for CR functions.

Theorem 28. *Suppose M is a hypersurface in $\mathbb{C}^n$ of class C^k with $3 \leq k \leq \infty$, and suppose p is a point in M where the Levi form has eigenvalues of opposite sign. Then each CR function on M that is a priori C^1 in a neighborhood of p must be of class C^k in a neighborhood of p. If in addition M is real analytic, then an a priori C^1 function defined near p must be a real analytic near p.*

Example 29. To illustrate the CR extension theorem, let us analyze the Heisenberg Group in the case $n = 2$. Let $M = \left\{(z, w) \in \mathbb{C}^2; \text{Im } z = |w|^2\right\}$. Note that $|w|^2 = w\overline{w}$. By Observation

10, $H^{1,0} = \mathbb{L}$ and $H^{0,1} = \overline{\mathbb{L}}$ are spanned by vector fields $\frac{\partial}{\partial w} + 2i\frac{\partial}{\partial z}\overline{w}$ and $\frac{\partial}{\partial \overline{w}} - 2i\frac{\partial}{\partial \overline{z}}w$, respectively. So

$$\begin{aligned}
[\overline{L}, L] &= \left(\frac{\partial}{\partial \overline{w}} - 2i\frac{\partial}{\partial \overline{z}}w\right)\left(\frac{\partial}{\partial w} + 2i\frac{\partial}{\partial z}\overline{w}\right) - \left(\frac{\partial}{\partial w} + 2i\frac{\partial}{\partial z}\overline{w}\right)\left(\frac{\partial}{\partial \overline{w}} - 2i\frac{\partial}{\partial \overline{z}}w\right) \\
&= \cancel{\frac{\partial^2}{\partial \overline{w}\partial w}} + \frac{\partial}{\partial w}\left(2i\frac{\partial}{\partial \overline{z}}w\right) - \left(2i\frac{\partial}{\partial \overline{z}}w\right)\frac{\partial}{\partial \overline{w}} - \cancel{4i^2 w\overline{w}\frac{\partial^2}{\partial \overline{z}\partial z}} - \cancel{\frac{\partial^2}{\partial w\partial \overline{w}}} \qquad (2) \\
&+ \frac{\partial}{\partial w}\left(2i\frac{\partial}{\partial \overline{z}}w\right) - \left(2i\frac{\partial}{\partial z}\overline{w}\right)\frac{\partial}{\partial \overline{w}} + \cancel{4i^2 w\overline{w}\frac{\partial^2}{\partial z\partial \overline{z}}}. \qquad (3)
\end{aligned}$$

So, we have

$$\begin{aligned}
[\overline{L}, L] &= 2i\frac{\partial}{\partial z} + \cancel{2i\overline{w}\frac{\partial^2}{\partial \overline{w}\partial z}} - \cancel{2iw\frac{\partial^2}{\partial \overline{z}\partial w}} + 2i\frac{\partial}{\partial \overline{z}} + \cancel{2iw\frac{\partial^2}{\partial w\partial \overline{z}}} - \cancel{2i\overline{w}\frac{\partial^2}{\partial z\partial \overline{w}}} \\
&= 2i\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\right) \neq 0.
\end{aligned}$$

Therefore, the Heisenberg Group in this case satisfies the CR extension theorem.

3. MAIN RESULT

We can now answer the question previously raised about CR extension. We have identified the necessary conditions such that, given a hypersurface M and a set ω in M, there exists an open set to which every CR function on ω extends holomorphically. The key factor is that the Levi form must have at least one positive or one negative eigenvalue for a one-sided extension, or at least one positive and one negative eigenvalue for a two-sided extension. Therefore, it is clear that for a Levi flat hypersurface, it is impossible to satisfy any of these conditions. Consequently, we understand why the hypersurface in Example 15 does not allow for a CR extension: it is due to being Levi flat.

This leads us to consider: how can we modify the manifold in Example 15 so that it satisfies the CR extension theorem? Furthermore, what are the conditions for a manifold to have non-zero eigenvalues in order to satisfy the CR extension theorem? To answer these questions, we first need to analyze the behaviors that differentiate the manifolds in Example 15 and Example 29, as this could not only help us understand the manifolds but also be a condition for satisfying the CR extension. Note that in Example 15, the manifold is Levi flat, so the Lie bracket between $\overline{\mathbb{L}}$ and $\mathbb{L}$ is automatically zero. In contrast, for Example 29, the Lie bracket between these vector fields is $2i\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\right)$. Since

$$2i\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\right) = 2i\left(1\cdot\frac{\partial}{\partial z} + 1\cdot\frac{\partial}{\partial \overline{z}}\right). \qquad (4)$$

Note that we can express 4 as

$$2i\left(\frac{\partial^2}{\partial \overline{w}\partial w}(w\overline{w})\frac{\partial}{\partial z} + \frac{\partial^2}{\partial w\partial \overline{w}}(\overline{w}w)\frac{\partial}{\partial \overline{z}}\right).$$

In the example 15, we have

$$2i\left(0\cdot\frac{\partial}{\partial z}+0\cdot\frac{\partial}{\partial \overline{z}}\right)=2i\left(\frac{\partial^2}{\partial \overline{w}\partial w}(0)\frac{\partial}{\partial z}+\frac{\partial^2}{\partial w\partial \overline{w}}(0)\frac{\partial}{\partial \overline{z}}\right). \tag{5}$$

Thus, the Equations 4 and 5 provide a framework for establishing the conditions and answers we seek.

These questions and previous analyses motivate the novel result on CR extension that we present in this paper, along with its proof.

Theorem 30 (Main result)**.** *Let* $M=\left\{(z,w)\in\mathbb{C}^2;\,\text{Im}\ z=\phi(w,\overline{w})\right\}$ *be a real hypersurface in* $\mathbb{C}^n$*, where* $\phi:\mathbb{C}^2\longrightarrow\mathbb{C}$ *is a polynomial function of at least degree* 2*. The hypersurface* M *has a CR Extension if* $\dfrac{\partial^2\overline{\phi}}{\partial w\partial\overline{w}}$ *and* $\dfrac{\partial^2\phi}{\partial\overline{w}\partial w}$ *are non-zero.*

Demonstração. We prove this theorem with induction over the degree of polynomial function.

For degree 2, let $\phi(w,\overline{w})=\displaystyle\sum_{l+k\leq 2}\alpha_{l,k}w^l\overline{w}^k$. By Observation 10, $\mathbb{L}$ and $\overline{\mathbb{L}}$ are spanned vector fields $\dfrac{\partial}{\partial w}+2i\left(\displaystyle\sum_{l+k\leq 2}l\alpha_{l,k}w^{l-1}\overline{w}^k\right)\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial\overline{w}}-2i\left(\displaystyle\sum_{l+k\leq 2}l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^k\right)\dfrac{\partial}{\partial\overline{z}}$, respectively.

Thus,

$$\begin{aligned}
[\overline{L},L] &= \left[\frac{\partial}{\partial\overline{w}}-2i\left(\sum_{l+k\leq 2}l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^k\right)\frac{\partial}{\partial\overline{z}}\right]\left[\frac{\partial}{\partial w}+2i\left(\sum_{l+k\leq 2}l\alpha_{l,k}w^{l-1}\overline{w}^k\right)\frac{\partial}{\partial z}\right]\\
&- \left[\frac{\partial}{\partial w}+2i\left(\sum_{l+k\leq 2}l\alpha_{l,k}w^{l-1}\overline{w}^k\right)\frac{\partial}{\partial z}\right]\left[\frac{\partial}{\partial\overline{w}}-2i\left(\sum_{l+k\leq 2}l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^k\right)\frac{\partial}{\partial\overline{z}}\right]\\
&= \cancel{\frac{\partial^2}{\partial\overline{w}\partial w}}+\frac{\partial}{\partial\overline{w}}\left[2i\left(\sum_{l+k\leq 2}l\alpha_{l,k}w^{l-1}\overline{w}^k\right)\frac{\partial}{\partial z}\right]-2i\left(\sum_{l+k\leq 2}l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^k\right)\frac{\partial^2}{\partial\overline{z}\partial w}\\
&- \cancel{4i^2\left(\sum_{l+k\leq 2}l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^k\right)\left(\sum_{l+k\leq 2}l\alpha_{l,k}w^{l-1}\overline{w}^k\right)}\frac{\partial^2}{\partial\overline{z}\partial z}-\cancel{\frac{\partial^2}{\partial w\partial\overline{w}}}\\
&+ \frac{\partial}{\partial w}\left[2i\left(\sum_{l+k\leq 2}l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^k\right)\frac{\partial}{\partial\overline{z}}\right]-2i\left(\sum_{l+k\leq 2}l\alpha_{l,k}w^{l-1}\overline{w}^k\right)\frac{\partial^2}{\partial z\partial\overline{w}}\\
&+ \cancel{4i^2\left(\sum_{l+k\leq 2}l\alpha_{l,k}w^{l-1}\overline{w}^k\right)\left(\sum_{l+k\leq 2}l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^k\right)\frac{\partial^2}{\partial z\partial\overline{z}}}\\
&= \frac{\partial}{\partial\overline{w}}\left[2i\left(\sum_{l+k\leq 2}l\alpha_{l,k}w^{l-1}\overline{w}^k\right)\frac{\partial}{\partial z}\right]-2i\left(\sum_{l+k\leq 2}l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^k\right)\frac{\partial^2}{\partial\overline{z}\partial w}\\
&+ \frac{\partial}{\partial w}\left[2i\left(\sum_{l+k\leq 2}l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^k\right)\frac{\partial}{\partial\overline{z}}\right]-2i\left(\sum_{l+k\leq 2}l\alpha_{l,k}w^{l-1}\overline{w}^k\right)\frac{\partial^2}{\partial z\partial\overline{w}}\\
&= 2i\left(\sum_{l+k\leq 2}lk\alpha_{l,k}w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z}+\cancel{2i\left(\sum_{l+k\leq 2}l\alpha_{l,k}w^{l-1}\overline{w}^k\right)\frac{\partial^2}{\partial\overline{w}\partial z}}
\end{aligned}$$

$$
\begin{aligned}
&- \; \cancel{2i\left(\sum_{l+k\leq 2} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial^2}{\partial\overline{z}\partial w}} + 2i\left(\sum_{l+k\leq 2} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial\overline{z}}\\
&+ \; \cancel{2i\left(\sum_{l+k\leq 2} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial^2}{\partial w\partial\overline{z}}} - \cancel{2i\left(\sum_{l+k\leq 2} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial^2}{\partial z\partial\overline{w}}}\\
&= \; 2i\left(\sum_{l+k\leq 2} lk\alpha_{l,k}w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z} + 2i\left(\sum_{l+k\leq 2} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial\overline{z}}\\
&= \; 2i\left[\left(\sum_{l+k\leq 2} lk\alpha_{l,k}w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z} + \left(\sum_{l+k\leq 2} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial\overline{z}}\right]\\
&= \; 2i\left(\frac{\partial^2\phi}{\partial\overline{w}\partial w}\frac{\partial}{\partial z} + \frac{\partial^2}{\partial w\partial\overline{w}}\frac{\partial}{\partial\overline{z}}\right)
\end{aligned}
$$

Suppose that holds for some $n \in \mathbb{N}$, let $\phi(w,\overline{w}) = \sum_{l+k\leq n} \alpha_{l,k}w^l\overline{w}^k$, so

$$
\begin{aligned}
[\overline{L}, L] \; &= \; \left(\frac{\partial}{\partial\overline{w}} - 2i\left(\sum_{l+k\leq n} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial}{\partial\overline{z}}\right)\left(\frac{\partial}{\partial w} + 2i\left(\sum_{l+k\leq n} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial}{\partial z}\right)\\
&- \; \left(\frac{\partial}{\partial w} + 2i\left(\sum_{l+k\leq n} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial\overline{w}} - 2i\left(\sum_{l+k\leq n} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial}{\partial\overline{z}}\right)\\
&= \; 2i\left[\left(\sum_{l+k\leq n} lk\alpha_{l,k}w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z} + \left(\sum_{l+k\leq n} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial\overline{z}}\right]\\
&= \; 2i\left(\frac{\partial^2\phi}{\partial\overline{w}\partial w}\frac{\partial}{\partial z} + \frac{\partial^2\overline{\phi}}{\partial w\partial\overline{w}}\frac{\partial}{\partial\overline{z}}\right).
\end{aligned}
$$

We need to prove that it holds for $n+1$. Let $\phi(w,\overline{w}) = \sum_{l+k\leq n+1} \alpha_{l,k}w^l\overline{w}^k$. Note that the generator for $\mathbb{L}$ and $\overline{\mathbb{L}}$ are, respectively, the vector fields

$$
\frac{\partial}{\partial w} + 2i\left(\sum_{l+k\leq n+1} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial}{\partial z} \text{ and}
$$

$$
\frac{\partial}{\partial\overline{w}} - 2i\left(\sum_{l+k\leq n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial}{\partial\overline{z}}.
$$

Using the bi-linearity of the Lie Bracket, we have

$$
\begin{aligned}
[\overline{L}, L] \; &= \; \left[\frac{\partial}{\partial\overline{w}} - 2i\left(\sum_{l+k\leq n} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial}{\partial\overline{z}}, \frac{\partial}{\partial w} + 2i\left(\sum_{l+k\leq n} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial}{\partial z}\right]\\
&+ \; \left[\frac{\partial}{\partial\overline{w}} - 2i\left(\sum_{l+k\leq n} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial}{\partial\overline{z}}, 2i\left(\sum_{l+k=n+1} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial}{\partial z}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \left[-2i\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial}{\partial\overline{z}}, \frac{\partial}{\partial w} + 2i\left(\sum_{l+k\leq n} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial}{\partial z}\right] \\
&+ \left[-2i\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial}{\partial\overline{z}}, 2i\left(\sum_{l+k=n+1} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial}{\partial z}\right] \\
&= 2i\left[\left(\sum_{l+k\leq n} lk\alpha_{l,k}w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z} + \left(\sum_{l+k\leq n} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial\overline{z}}\right] \\
&+ \frac{\partial}{\partial\overline{w}}\left[2i\left(\sum_{l+k=n+1} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial}{\partial z}\right] \\
&- 4i^2\left(\sum_{l+k\leq n} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\left(\sum_{l+k=n+1} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial^2}{\partial\overline{z}\partial z} \\
&- 2i\left(\sum_{l+k=n+1} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial^2}{\partial z\partial\overline{w}} \\
&+ 4i^2\left(\sum_{l+k=n+1} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\left(\sum_{l+k\leq n} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial^2}{\partial z\partial\overline{z}} \\
&- 2i\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial^2}{\partial\overline{z}\partial w} \\
&- 4i^2\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\left(\sum_{l+k\leq n} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial^2}{\partial\overline{z}\partial z} \\
&+ \frac{\partial}{\partial w}\left[2i\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial}{\partial\overline{z}}\right] \\
&+ 4i^2\left(\sum_{l+k\leq n} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right) \\
&- 4i^2\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\left(\sum_{l+k=n+1} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial^2}{\partial\overline{z}\partial z} \\
&+ 4i^2\left(\sum_{l+k=n+1} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial^2}{\partial z\partial\overline{z}} \\
&= 2i\left[\left(\sum_{l+k\leq n} lk\alpha_{l,k}w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z} + \left(\sum_{l+k\leq n} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial\overline{z}}\right] \\
&+ \frac{\partial}{\partial\overline{w}}\left[2i\left(\sum_{l+k=n+1} l\alpha_{l,k}w^{l-1}\overline{w}^{k}\right)\frac{\partial}{\partial z}\right]
\end{aligned}
$$

$$
\begin{aligned}
& - \; 2i\left(\sum_{l+k=n+1} l\alpha_{l,k} w^{l-1}\overline{w}^{k}\right)\frac{\partial^2}{\partial z\partial \overline{w}} - 2i\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial^2}{\partial \overline{z}\partial w} \\
& + \; \frac{\partial}{\partial w}\left[2i\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial}{\partial \overline{z}}\right] \\
& = \; 2i\left[\left(\sum_{l+k\leq n} lk\alpha_{l,k} w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z} + \left(\sum_{l+k\leq n} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial \overline{z}}\right] \\
& + \; 2i\left(\sum_{l+k=n+1} lk\alpha_{l,k} w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z} \\
& + \; \cancel{2i\left(\sum_{l+k=n+1} l\alpha_{l,k} w^{l-1}\overline{w}^{k}\right)\frac{\partial^2}{\partial \overline{w}\partial z}} - \cancel{2i\left(\sum_{l+k=n+1} l\alpha_{l,k} w^{l-1}\overline{w}^{k}\right)\frac{\partial^2}{\partial z\partial \overline{w}}} \\
& - \; \cancel{2i\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial^2}{\partial \overline{z}\partial w}} \\
& + \; 2i\left(\sum_{l+k=n+1} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial \overline{z}} + \cancel{2i\left(\sum_{l+k=n+1} l\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k}\right)\frac{\partial^2}{\partial w\partial \overline{z}}} \\
& = \; 2i\left[\left(\sum_{l+k\leq n} lk\alpha_{l,k} w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z} + \left(\sum_{l+k\leq n} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial \overline{z}}\right] \\
& + \; 2i\left(\sum_{l+k=n+1} lk\alpha_{l,k} w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z} \\
& + \; 2i\left(\sum_{l+k=n+1} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial \overline{z}} \\
& = \; 2i\left[\left(\sum_{l+k\leq n+1} lk\alpha_{l,k} w^{l-1}\overline{w}^{k-1}\right)\frac{\partial}{\partial z} + \left(\sum_{l+k\leq n+1} lk\overline{\alpha}_{l,k}\overline{w}^{l-1}w^{k-1}\right)\frac{\partial}{\partial \overline{z}}\right] \\
& = \; 2i\left(\frac{\partial^2\phi}{\partial\overline{w}\partial w}\frac{\partial}{\partial z} + \frac{\partial^2\overline{\phi}}{\partial w\partial\overline{w}}\frac{\partial}{\partial \overline{z}}\right)
\end{aligned}
$$

Therefore, the theorem is proven. Thus, we are able to answer the previously raised questions as well as analyze other examples that may or may not satisfy the extension conditions according to the theorem. □

Example 31. Suppose $M = \{(z,w) \in \mathbb{C}^2; \operatorname{Im} z = \phi(w,\overline{w})\}$. By Theorem 30, M does not have CR extension if ϕ is a first-degree polynomial function or a constant function. Note that in Example 15, ϕ is the identically zero function (constant function) and does not satisfy the theorem.

4. CONCLUSIONS

The present study addressed the problem of holomorphic extension of CR functions defined on a specific class of hypersurfaces in $\mathbb{C}^2$. By imposing conditions on the defining function of the hypersurface and employing Hans Lewy's Extension Theorem as the principal analytical tool, we derived sufficient criteria under which the extension exists. The arguments developed throughout the work emphasize the interplay between the analytical properties of the Cauchy–Riemann system and the geometric structure encoded in the Levi form associated with the hypersurface.

A central aspect of our analysis lies in the role of the mixed second derivatives of the defining function $\phi(w, \overline{w})$. It was established that when these derivatives do not vanish, the hypersurface allows a holomorphic extension of CR functions. This result provides a geometric characterization of the necessary and sufficient conditions for the extension phenomenon and includes, as particular cases, several classical results described in the literature. The approach followed here also clarifies how the nondegeneracy of the Levi form ensures the propagation of analyticity across the real hypersurface.

Beyond its intrinsic theoretical interest, this investigation highlights methodological aspects that may be extended to broader contexts. In particular, the techniques employed in this study, combining analytic continuation and differential–geometric arguments, may serve as a foundation for further exploration in higher dimensions, where the complexity of the CR structure increases substantially. Moreover, these methods can be adapted to settings where the defining functions are not polynomial, thereby offering potential for the generalization of the extension theorems to wider classes of manifolds.

Future research may focus on extending the present results to real hypersurfaces of higher codimension, on analyzing the regularity of the extended functions in the boundary, and on identifying geometric invariants capable of characterizing the precise conditions under which holomorphic extension persists. The investigation of these aspects would contribute to a deeper understanding of CR geometry and its connections with the theory of partial differential equations in several complex variables.

More details of CR functions and its extensions can be found at (Berhanu; Cordaro; Hounie, 2008), (Boggess, 1991), (Lewy, 1956) (Hörmander, 1973), (Hörmander, 1983), (Hounie, 1979) and (Silva, 2022).

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