RMAT
ISSN: 2237-8103
REVISTA DE MATEMÁTICA DA UFOP
OPEN ACCESS
Volume 1, Número 1, 2026
https://doi.org/10.63801/rmat.v1i1.8528
Recebido: 16/03/2026
Publicado: 01/06/2026

Coupled Harmonic Oscillators Model for Financial Time Series

Modelo de Osciladores Harmônicos Acoplados para Séries Temporais Financeiras
Modelo de Osciladores Armónicos Acoplados para Series Temporales Financieras

Monique Rafaella Anunciação de Oliveira
<monique@ufop.edu.br>
Universidade Federal de Ouro Preto, Ouro Preto, MG, Brasil
<https://orcid.org/0000-0002-1010-9910>

Lucélia Viviane Vaz Raad
<lucelia@cefetmg.br>
Centro Federal de Educação Tecnológica de Minas Gerais, Belo Horizonte, MG, Brasil
<https://orcid.org/0000-0003-2888-0823>

A. R. Bosco de Magalhães
<magalhaes@cefetmg.br>
Centro Federal de Educação Tecnológica de Minas Gerais, Belo Horizonte, MG, Brasil
<https://orcid.org/0000-0002-5324-1444>

Abstract
In this paper, we present a model based on linearly coupled dissipative harmonic oscillators to characterize asset price dynamics, whose fundamental elements are restoring forces, inertia, and damping. The dynamic variables of the physical system are replaced with logarithmic differences between asset prices and reference prices calculated via moving averages. The model is applied to stocks comprising the Brazilian index Ibovespa. The estimated damping coefficients and spring constants suggest attractive and repulsive forces in the market, with a tendency towards regression to the mean and damped joint market movements. The distributions of these parameters indicate significant variations in the degree of interconnection among the stocks, offering valuable insights for asset selection in hedging strategies. Quantities analogous to kinetic and potential energies display power-law-tailed distributions and multifractal behavior, and their dynamics reveal signs of events that have impacted both domestic and international markets.

Keywords: *Econophysics. Harmonic oscillator network. Financial market model. Heavy-tailed distribution. Multifractal time series.*

Resumo
Neste artigo, apresentamos um modelo baseado em osciladores harmônicos dissipativos linearmente acoplados para caracterizar a dinâmica dos preços de ativos, cujos elementos fundamentais são forças restauradoras, inércia e amortecimento. As variáveis dinâmicas do sistema físico são substituídas por diferenças logarítmicas entre os preços dos ativos e preços de referência calculados por meio de médias móveis. O modelo é aplicado à ações que compõem o índice brasileiro Ibovespa. Os coeficientes de amortecimento e as constantes elásticas estimados sugerem a presença de forças atrativas e repulsivas no mercado, com tendência à regressão à média e a movimentos amortecidos conjuntos do mercado. As distribuições desses parâmetros indicam variações significativas no grau de interconexão entre as ações, oferecendo percepções valiosas para a seleção de ativos em estratégias de hedge. Quantidades análogas às energias cinética e potencial exibem distribuições com caudas em lei de potência e comportamento multifractal, e suas dinâmicas revelam indícios de eventos que impactaram tanto os mercados domésticos quanto os internacionais.

Palavras-chave: *Econofísica. Rede de osciladores harmônicos. Modelo de mercado financeiro. Distribuição de cauda pesada. Séries temporais multifractais.*

Resumen
En este artículo presentamos un modelo basado en osciladores armónicos disipativos acoplados linealmente para caracterizar la dinámica de los precios de los activos, cuyos elementos fundamentales son fuerzas restauradoras, inercia y amortiguamiento. Las variables dinámicas del sistema físico se sustituyen por diferencias logarítmicas entre los precios de los activos y precios de referencia calculados mediante medias móviles. El modelo se aplica a las acciones que componen el índice brasileño Ibovespa. Los coeficientes de amortiguamiento y las constantes elásticas estimados sugieren la presencia de fuerzas atractivas y repulsivas en el mercado, con una tendencia a la regresión a la media y a movimientos conjuntos amortiguados del mercado. Las distribuciones de estos parámetros indican variaciones significativas en el grado de interconexión entre las acciones, ofreciendo perspectivas valiosas para la selección de activos en estrategias de cobertura (hedge). Magnitudes análogas a las energías cinética y potencial exhiben distribuciones con colas de ley de potencia y comportamiento multifractal, y sus dinámicas revelan indicios de eventos que han impactado tanto los mercados domésticos como los internacionales.

Palabras-Clave: *Econofísica. Red de osciladores armónicos. Modelo de mercado financiero. Distribución de cola pesada. Series temporales multifractales.*

1. INTRODUCTION

A wide range of physical systems are modeled through harmonic oscillators. Classic examples are the spring-mass system and the electrical circuit composed of an inductor and a capacitor. When small oscillations are assumed, harmonic oscillators constitute linear approximations in many situations; the simple pendulum is the most explored example. In classical formalism, it is usual to include damping straightforwardly through forces proportional to velocity. In quantum physics, more elaborate strategies are needed; damping can be added to the model, for example, through oscillators coupled to the system of interest (Ford; Kac; Mazur, 1965; Caldeira; Leggett, 1983; Bosco de Magalhães *et al.*, 2004). Networks of linearly coupled harmonic oscillators are relatively simple systems to deal with; nevertheless, they exhibit interesting phenomena. Some examples are synchronization (Strogatz, 2000; Dörfler; Bullo, 2014) and the emergence of decoherence-free subspaces (Agarwal, 1974; Lidar; Whaley, 2003; Bosco de Magalhães, 2011).

In 1933, Ragnar Frisch introduced a damped oscillator-based model for price dynamics (Frisch, 1933). He considered that major movements in the economy generate oscillations that are damped by transaction costs. The fact that oscillations vanish is not observed in empirical data led Frisch to the idea of a noise-driven model. This essay contributed to the development of the real business cycle (RBC) and the dynamic stochastic general equilibrium (DSGE) methods for analyzing business cycles; both approaches have developed macrodynamic models with exogenous and random shocks and with the use of calibration. Recently, classical and quantum harmonic oscillators have been explored in econophysics. A quadratic potential whose center is calculated as a moving average of past prices is the core of the potentials of unbalanced complex kinetics (PUCK) random walk model for market price (Takayasu; Mizuno; Takayasu, 2006). In this model, prices can be attracted or repulsed by a reference price. A simple harmonic oscillator with noise composes a model for the centred return on the London Stock Exchange's FTSE All Share Index (Ataullah; Tippett, 2007). In (Ye; Huang, 2008), the quantum harmonic oscillator is utilized in a model in which the market works as a measuring apparatus for the asset value (represented by a wave packet) and produces the price as a result. A financial Schrödinger equation derived through the extremization of Fisher information is introduced in (Nastasiuk, 2015): the quantum particle in parabolic potential well corresponds to market efficiency. A network of damped harmonic oscillators provided a model for price series in local real estate markets in three cities in Poland (Kulesza; Belej, 2015). A quantum spatial-periodic harmonic oscillator potential well was employed to investigate stock dynamics in daily price-limited markets in China (Meng *et al.*, 2015). In (Meng; Zhang; Guo, 2016), single stocks and the corresponding index are mapped into a reservoir of quantum harmonic oscillators and the system of interest, respectively; deviations from rationality are associated with the Planck constant. The agent-based financial market model proposed in (Xavier; Atman; Bosco de Magalhães, 2017) leads to harmonic oscillator price dynamics; depending on the parameters that define the interactions between the agents, the underdamped, critically damped, or overdamped regime is produced. In (Ahn *et al.*, 2017), a model for return distribution based on the quantum harmonic oscillator is proposed and compared to traditional stochastic process models. The quantum harmonic oscillator is utilized in a model where supply and demand are represented by a probabilistic wave (Orrell, 2020). In (Garcia *et al.*, 2020), a price trend forecast model inspired by the damped harmonic oscillator is introduced and tested with Brazilian stocks.

In this contribution, we present a model for market dynamics based on dissipative harmonic oscillators featuring position and velocity couplings. Characteristics inherent to physical systems, such as inertia, restoration of equilibrium position, and damping, are linked to key concepts in time series modeling, including mean reversion and stationarity (Enders, 2015). The model is described in detail in the next section, where we also define, for the economic system, quantities analogous to kinetic and potential energy. In Section 3, we apply the model to the daily closing price series of 65 Brazilian stocks relative to the interval from 12/17/2004 to 05/29/2020. We delve into the connection between the evolution of the system energies and the market events that occurred during the analyzed period. The distributions of the model parameters are examined to provide insights into the interrelationships between the assets. Heavy tails are evident in these distributions, as well as in the distributions associated with quantities analogous to energies. The multifractal behavior of the kinetic and potential energy series is revealed. In Section 4, we present our final remarks.

2. MODEL

Let us consider a system composed of n linearly coupled oscillators whose dynamics is described by the differential equations

$$
\begin{aligned}
M_1 X_1'' &= -k_1 X_1 + \sum_{j\neq 1} k_{1,j}\left(X_j - X_1\right) \\
&\quad -\gamma_1 X_1' + \sum_{j\neq 1} \gamma_{1,j}\left(X_j' - X_1'\right), \\
M_2 X_2'' &= -k_2 X_2 + \sum_{j\neq 2} k_{2,j}\left(X_j - X_2\right) \\
&\quad -\gamma_2 X_2' + \sum_{j\neq 2} \gamma_{2,j}\left(X_j' - X_2'\right), \\
&\vdots \\
M_n X_n'' &= -k_n X_n + \sum_{j\neq n} k_{n,j}\left(X_j - X_n\right) \\
&\quad -\gamma_n X_n' + \sum_{j\neq n} \gamma_{n,j}\left(X_j' - X_n'\right),
\end{aligned}
\tag{1}
$$

where X_i represents the position of oscillator i ($i = 1, 2, \ldots, n$), and the derivatives are taken with respect to time t. Each oscillator i is associated with a mass M_i, a spring constant k_i, and a viscous damping coefficient γ_i. Position and velocity couplings are described through the constants $k_{i,j}$ and $\gamma_{i,j}$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$), respectively.

We propose a model for financial market dynamics based on this physical system. It deals with n assets. To fulfill the role of the oscillator position, we take the logarithmic difference between the price $P_i(t)$ of asset i at time t and a reference price $\bar{P}_i(t)$ defined as a moving average in the parameter S:

$$X_i(t) = \ln P_i(t) - \ln \bar{P}_i(t), \tag{2}$$

where

$$\bar{P}_i(t) = \frac{1}{S} \sum_{\tau=t-S}^{t-1} P_i(\tau). \tag{3}$$

Such data processing is similar to that performed in (Resende *et al.*, 2017); it takes the attention away from the asset price and directs the focus to the deviation from a reference price. It also permits us to treat prices on different scales together. As it resembles the computation of the first difference, this transformation also increases the chances of stationarity of the series.

The damped oscillator evolution is characterized by inertia, damping, and restoring forces. Such components are analogous to concepts from finance. The emergence of financial bubbles and their bursting, which are consequences of the herd effect, brings an element of inertia to the financial dynamics (Tsay, 2010). The stationarity of a time series is related to the damping of oscillations arising from new information that reaches the market (Granger; Newbold, 1974). Restoring forces can be associated with mean reversion, which refers to the tendency of the price of an asset to converge to a reference value based on its fundamentals (Campbell; Lo; MacKinlay, 1997).

Employing definitions

$$\mathbf{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix}, \tag{4}$$

we write system (1) in matrix form:

$$\frac{d^2\mathbf{X}}{dt}(t) = \mathbf{B}\frac{d\mathbf{X}}{dt}(t) + \mathbf{C}\mathbf{X}(t). \tag{5}$$

The following relationships hold:

$$\gamma_i = -M_i \sum_{j=1}^{n} b_{i,j}, \qquad k_i = -M_i \sum_{j=1}^{n} c_{i,j}, \tag{6}$$

and

$$\gamma_{i,j} = M_i b_{i,j}, \qquad k_{i,j} = M_i c_{i,j}, \qquad \text{for } i \neq j. \tag{7}$$

Eq. (5) can be discretized as

$$\frac{\mathbf{X}(t+\Delta t) - 2\mathbf{X}(t) + \mathbf{X}(t-\Delta t)}{(\Delta t)^2} = \mathbf{B}\frac{\mathbf{X}(t+\Delta t) - \mathbf{X}(t-\Delta t)}{2\Delta t} + \mathbf{C}\mathbf{X}(t). \tag{8}$$

Taking $\Delta t = 1$ and defining $\tau = t + 1$, we get

$$\mathbf{X}(\tau) = \mathbf{P}\mathbf{X}(\tau - 1) + \mathbf{Q}\mathbf{X}(\tau - 2), \tag{9}$$

where

$$\begin{aligned} \mathbf{P} &= \left(\mathbf{I} - \frac{1}{2}\mathbf{B}\right)^{-1} (2\mathbf{I} + \mathbf{C}), \\ \mathbf{Q} &= -\left(\mathbf{I} - \frac{1}{2}\mathbf{B}\right)^{-1} \left(\mathbf{I} + \frac{1}{2}\mathbf{B}\right), \end{aligned} \tag{10}$$

and $\mathbf{I}$ is an $n \times n$ identity matrix. This system is characterized by a second-order vector autoregressive model (VAR):

$$X_i(t) = \sum_{j=1}^{n} \left(p_{i,j} X_j(t-1) + q_{i,j} X_j(t-2)\right) + \varepsilon_i(t), \tag{11}$$

for $i = 1, 2, \cdots, n$, where $p_{i,j}$ and $q_{i,j}$ are elements of $\mathbf{P}$ and $\mathbf{Q}$, respectively, and $\varepsilon_i(t)$ are residuals.

From the estimated values for p_{ij} and q_{ij}, we can find $b_{i,j}$ and $c_{i,j}$ by means of

$$\begin{aligned} \mathbf{B} &= 2(\mathbf{Q}+\mathbf{I})(\mathbf{Q}-\mathbf{I})^{-1}, \\ \mathbf{C} &= \mathbf{P}-(\mathbf{Q}+\mathbf{I})(\mathbf{Q}-\mathbf{I})^{-1}\mathbf{P}-2\mathbf{I}. \end{aligned} \tag{12}$$

Set $\{b_{i,j}, c_{i,j}\}$ has $2n^2$ elements, and set $\{M_i, k_i, \gamma_i, k_{i,j}, \gamma_{i,j}\}$, $2n^2+n$ elements. To balance the number of elements of the latter with that of the former, we take all masses $M_i = 1$. The remaining parameters, k_i, γ_i, $k_{i,j}$ and $\gamma_{i,j}$, are computed from $b_{i,j}$ and $c_{i,j}$ by using relations (6) and (7). For physical oscillators composed of masses, springs, and dampers, the spring constants and viscous damping coefficients are always positive: negative values would lead to forces in the opposite direction of what is physically expected. In those physical systems, the force exerted through springs and dampers by oscillator i on oscillator j and that exerted by oscillator j on oscillator i have the same absolute value and opposite directions. This leads to $k_{i,j} = k_{j,i}$ and $\gamma_{i,j} = \gamma_{j,i}$. For the model aimed at the financial market, such restrictions are not necessary. Anticorrelated asset movements can lead to negative $k_{i,j}$ and $\gamma_{i,j}$. The parameters $k_{i,j}$ and $\gamma_{i,j}$ describe the influence of stock j on stock i: if this influence is non-symmetrical, it is expected to find $k_{i,j} \neq k_{j,i}$ and $\gamma_{i,j} \neq \gamma_{j,i}$. Despite the possibility of such negative values and asymmetries, we continue to call k_i and $k_{i,j}$ spring constants, and γ_i and $\gamma_{i,j}$ damping coefficients, targeting the fluidity of language.

To explore the dynamics of the aggregated economic system, we define quantities analogous to potential and kinetic energies. Due to the possible differences between $k_{i,j}$ and $k_{j,i}$, we can not define a potential energy for the whole system associated with the dynamics described in equations (1). Thus, we start by defining quantities analogous to energies for each oscillator separately. The quantity analogous to the potential energy of oscillator i is defined as

$$U_i(t) = \frac{k_i X_i^2(t)}{2} + \sum_{j \neq i} \frac{k_{i,j}}{2} \left(X_i(t) - X_j(t)\right)^2, \tag{13}$$

and the quantity analogous to the kinetic energy of oscillator i, as

$$K_i(t) = \frac{1}{2} M_i \left(X_i'(t)\right)^2, \tag{14}$$

where

$$X_i'(t) = \frac{X_i(t+1) - X_i(t-1)}{2}. \tag{15}$$

Quantities connected with the total kinetic and potential energies are then defined as

$$K(t) = \sum_{i=1}^{n} K_i(t), \qquad U(t) = \sum_{i=1}^{n} U_i(t), \tag{16}$$

respectively.

3. APPLICATION TO THE BRAZILIAN MARKET

The model was applied to 65 stocks that make up the Brazilian Ibovespa index. Closing prices from 12/17/2004 to 05/29/2020 were considered, totaling $4,031$ points in each series.

The stocks chosen are assets that exist throughout the investigated period and are part of the index at some time in the interval. We took the parameter that defines the reference price as $S = 12$. Dickey-Fuller test was utilized to verify if the transformation described in (2) guaranteed the stationarity of the $X_i(t)$ series; for all assets, we rejected the null hypothesis of unit root with a confidence level of 95%, which allowed us to conclude that all series are stationary. [1]To analyze the residuals arising from the application of the model in these series (Eq. (11)), we used the Breusch-Godfrey test for serially correlated errors, and we rejected the null hypothesis of serial correlation with a significance level of 95%; thus, we conclude that such residuals are uncorrelated.

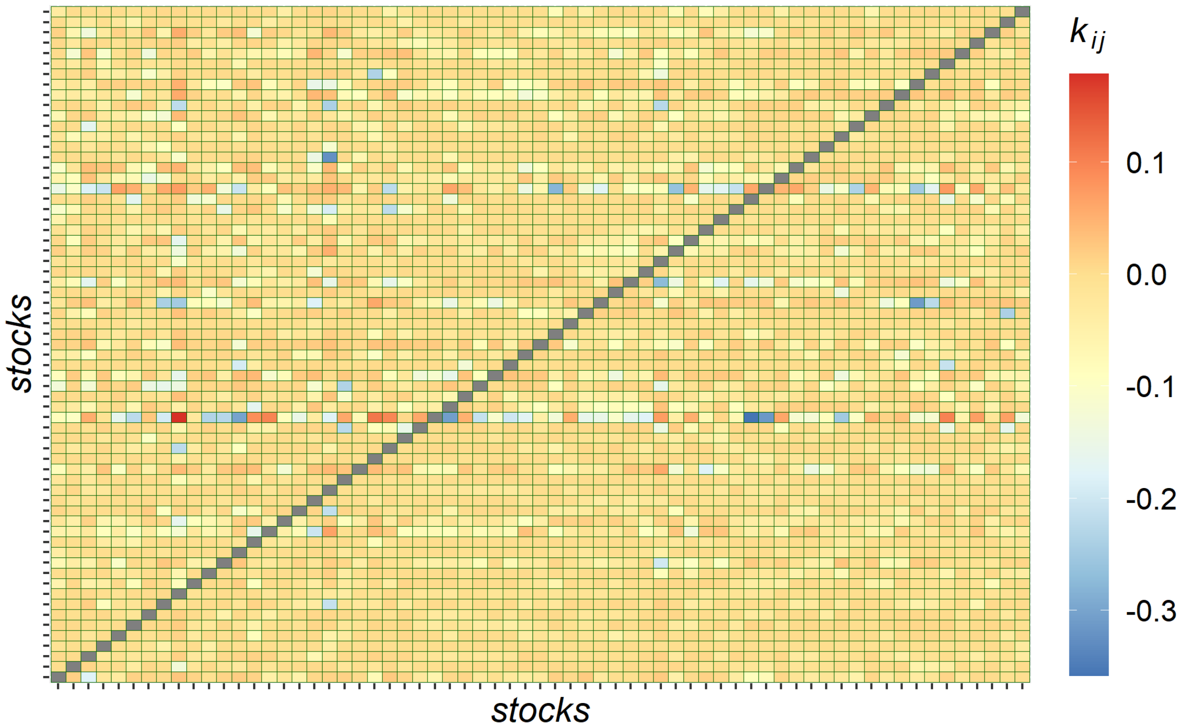

Figura 1. Spring constants $k_{i,j}$. Positions on the secondary diagonal have no values as $k_{i,j}$ is not defined for $i = j$.

From the VAR model estimated, we considered the statistically significant coefficients with a significance level of 5%, and calculated the spring constants and damping coefficients. An overview is provided in the heat maps shown in figures 1 and 2. All spring constants k_i are positive, with a mean of 2.92 and a standard deviation of 0.295. In the set $\{k_{i,j}\}$, we observe $2,281$ positive values of $k_{i,j}$ with a mean of 8.22×10^{-3} and a standard deviation of 1.12×10^{-2}. Additionally, there are $1,879$ negative values of $k_{i,j}$ with a mean of -4.23×10^{-2} and a standard deviation of 4.74×10^{-2}. The parameters k_i determine the restoring forces that drive the reversion to the mean. The fact that all k_i are positive indicates the significance of this effect in the analyzed series. The $k_{i,j}$ parameters are associated with the effects of simultaneous movements of stocks. When positive, they are linked to attractive forces, while negative values are related to repulsive forces. A large positive value of $k_{i,j}$ indicates that two stocks tend to

1 The stationarity of each series composing the VAR is necessary to estimate its coefficients. For more details, refer to (Hamilton, 1994)

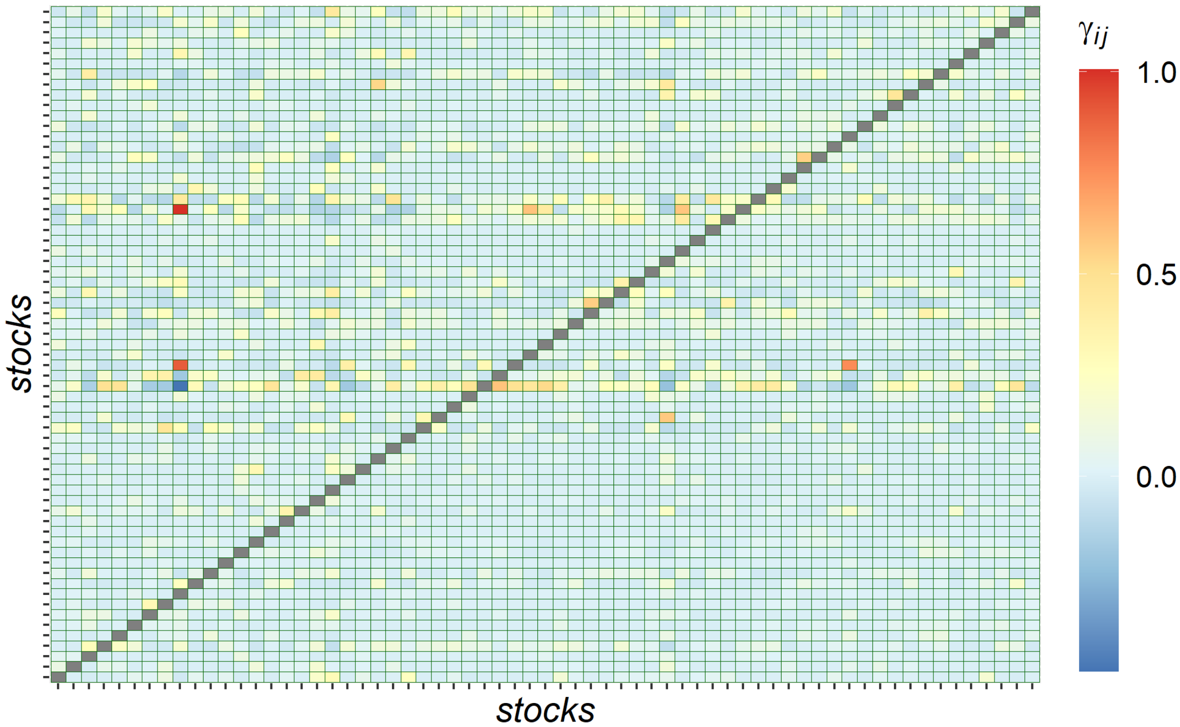

Figura 2. Damping coefficients $\gamma_{i,j}$. Positions on the secondary diagonal have no values as $\gamma_{i,j}$ is not defined for $i = j$.

move in the same direction, which implies that such assets should not be used in a potential hedging strategy. The prevalence of positive $k_{i,j}$ values suggests that attraction is more relevant than repulsion in the observed data, reflecting the tendency of the market to move collectively. There are 42 positive values of γ_i, with a mean of 0.36 and a standard deviation of 0.25, which is almost twice the number of negative values. There are 23 negative values of γ_i, with a mean of -0.41 and a standard deviation of 0.52. Regarding $\gamma_{i,j}$, there are $1,858$ positive values (with a mean of 8.24×10^{-2} and a standard deviation of 9.65×10^{-2}), and $2,302$ negative values (with a mean of -1.49×10^{-2} and a standard deviation of 2.37×10^{-2}). Both γ_i and $\gamma_{i,j}$ exhibit a predominance of positive values: considering positive and negative values, the means of both sets are positive. Positive values of γ_i correspond to the damping of movements of asset i relative to its equilibrium position. Therefore, in the analyzed series, there is a tendency for price movements relative to the reference price to be damped. The prevalence of positive values in the set of $\gamma_{i,j}$ indicates a tendency for relative movements between different assets to be damped, thereby suggesting stabilization towards collective movements.

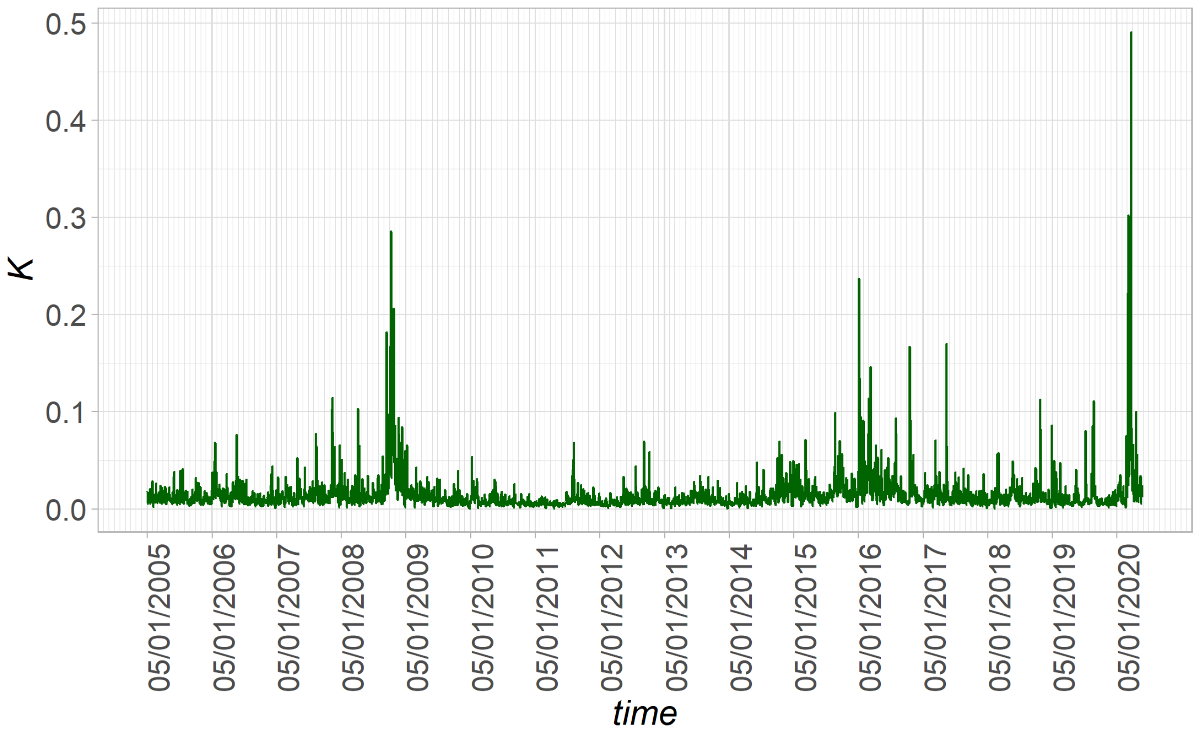

Figura 3. Time evolution of the quantity analogous to kinetic energy $K(t)$.

Figures 3 and 4 show the evolution of $K(t)$ and $U(t)$, respectively. The quantity $K(t)$ is nonnegative, by definition. The series $U(t)$ found is also nonnegative; this result can be justified by the fully positive k_i and by the predominance of positive $k_{i,j}$. Connections are observed between events that influence the financial market and the behavior of $K(t)$ and $U(t)$. The subprime financial crisis (Sanders, 2008) arises as a region with high values in the second half of 2008, with an outstanding peak in October. Another region with large movements shows up in 2020, related to the COVID-19 pandemic, declared by the World Health Organization on March 11 of that year (Cucinotta; Vanelli, 2020). The Brazilian domestic scenario also affects the

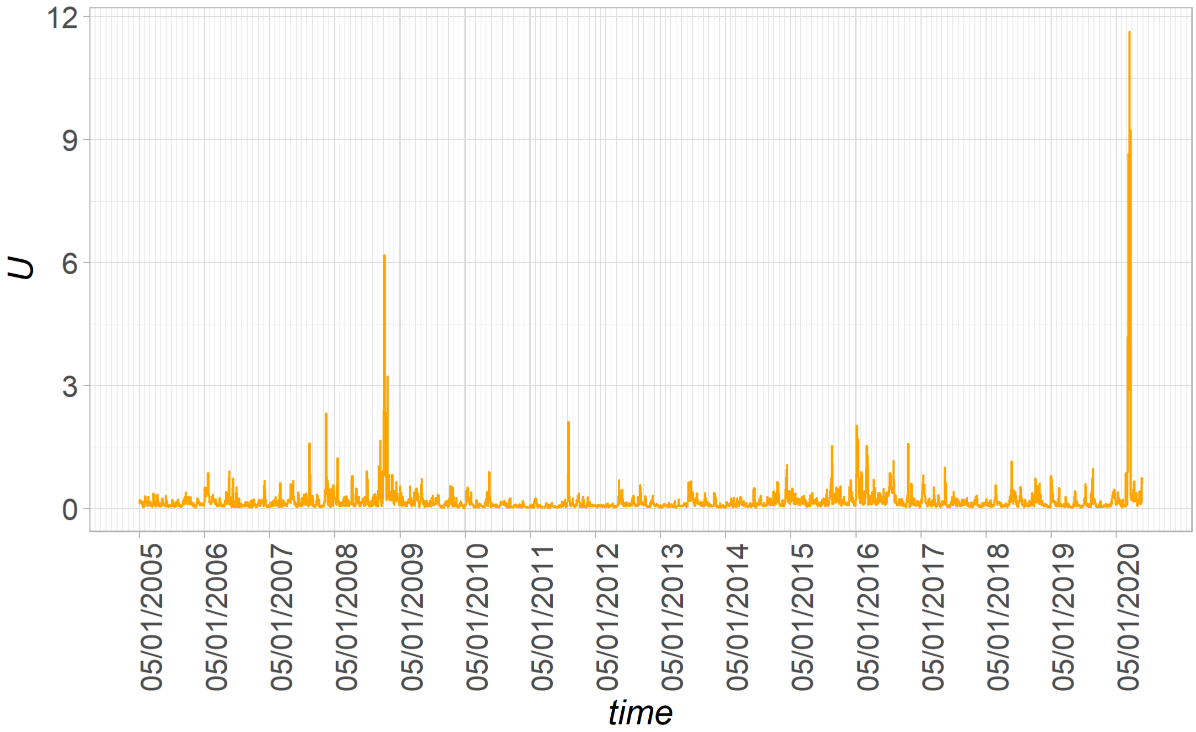

Figura 4. Time evolution of the quantity analogous to potential energy $U\ (t)$.

dynamics. In January 2016, the country's official inflation for the previous year was released, the highest in the last 13 years (Modenesi; Luporini; Pimentel, 2017). The political and economic turbulence that led to the impeachment process of President Dilma Rousseff, started on May 12, 2016, in the Senate and concluded on August 31, 2016, in the Chamber of Deputies (Avritzer, 2017), corresponds to the clustering of high values, observed mainly for $K\ (t)$, in this year. In October 2016, the former president of the Chamber of Deputies was arrested after losing his parliamentary immunity (Nunes; Melo, 2017), and on May 17, 2017, there was a scandal involving meat sector businessmen and high-ranking politicians (Silvestre *et al.*, 2018).

3.1. Stylized facts

Heavy tailed distributions and fractal time series are widely observed in financial data, both real and simulated (Mantegna; Stanley, 1999; Lux; Marchesi, 1999; Gopikrishnan *et al.*, 1999; Cajueiro; Tabak, 2004; Matteo; Aste; Dacorogna, 2005; Atman A. P. F. Amin Gonçalves, 2012; Kwapień; Drożdż, 2012; Yang *et al.*, 2015; Botta *et al.*, 2015; Buonocore; Aste; Matteo, 2016; Bosco de Magalhães, 2023). Such phenomena may be associated with complex relationships among economic agents (Ducha; Atman; Bosco de Magalhães, 2021). We demonstrate that the distributions of the parameters $k_{i,j}$ and $\gamma_{i,j}$, which describe the relationships between the dynamic variables associated with asset prices, display heavy tails. Similarly, heavy tails are evident in the distributions of potential and kinetic energy series. Moreover, multifractal behavior is explicitly manifested in these series.

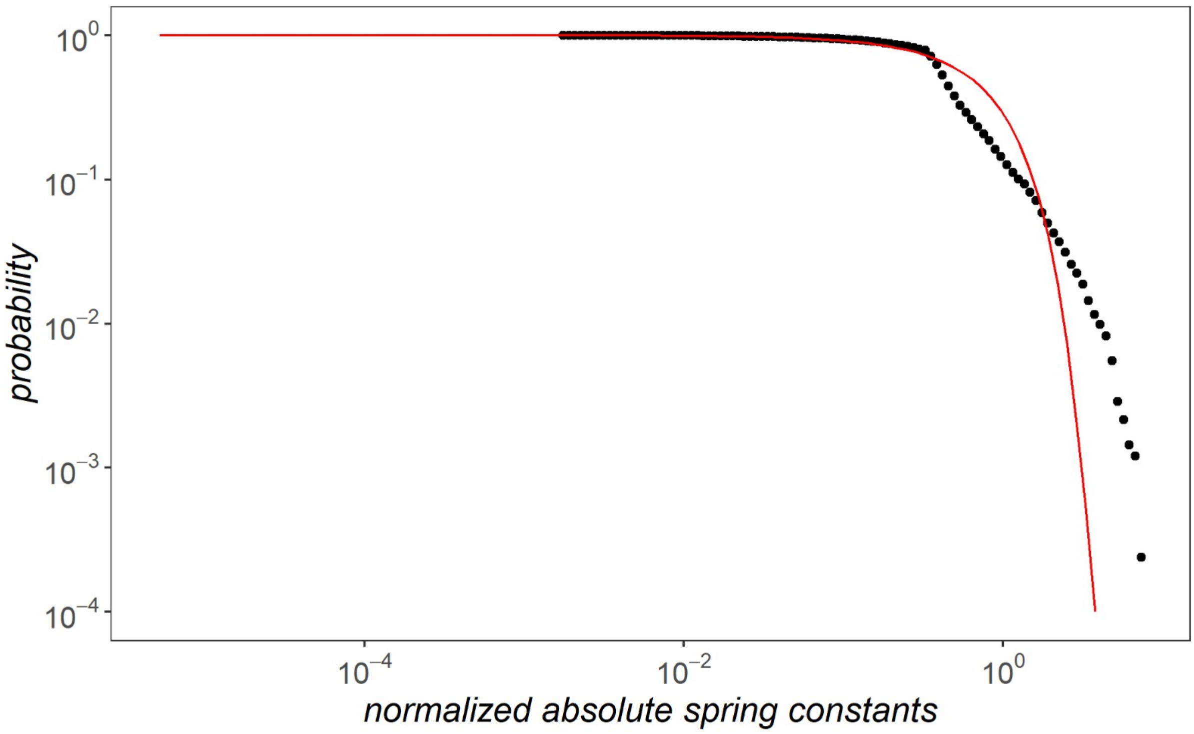

Figura 5. Dots represent the inverse cumulative distribution function of the absolute values of the normalized spring constants. For comparison, the red curve gives the Gaussian profile.

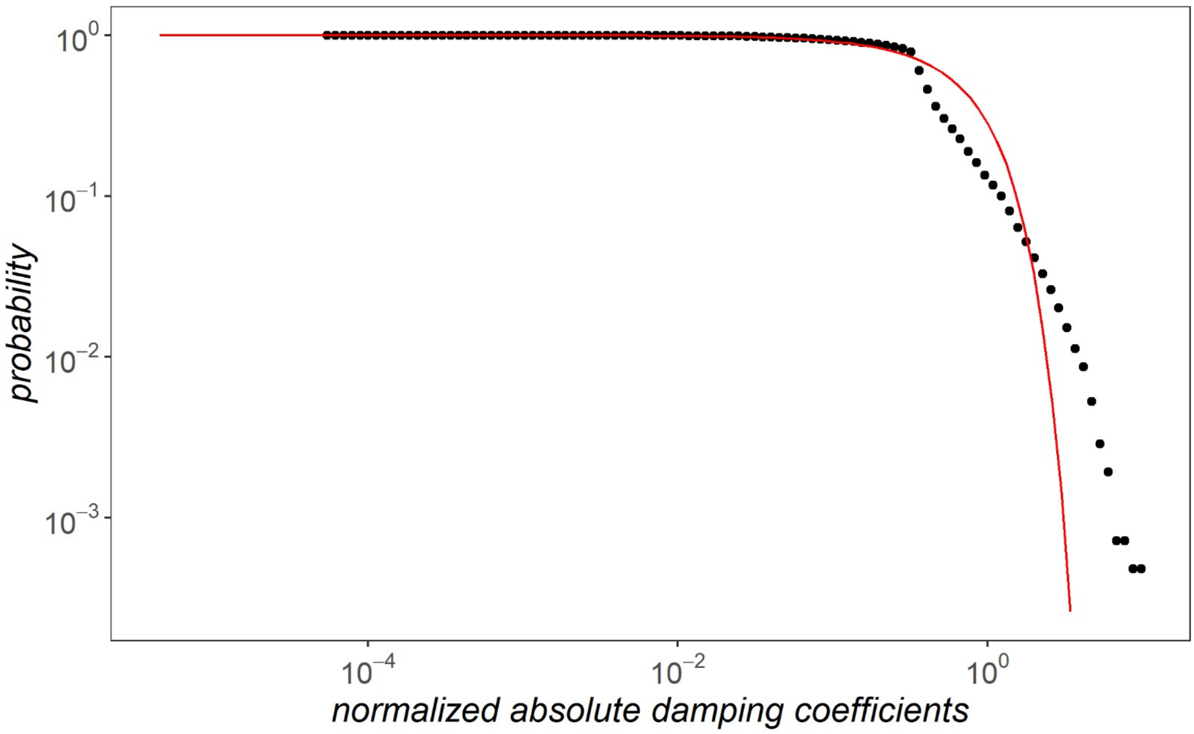

Figura 6. Dots represent the inverse cumulative distribution function of the absolute values of the normalized damping coefficients. For comparison, the red curve gives the Gaussian profile.

Normalized $k_{i,j}$ were calculated by subtracting the mean of the set $\{k_{i,j}\}$ from each element and dividing the result by the respective standard deviation. Similarly, we computed normalized $\gamma_{i,j}$ values using the same method. Fig. 5 shows the cumulative inverse distribution of the absolute values of the normalized spring constants. The difference between this distribution and the normal distribution is made explicit in the figure. The goodness-of-fit test shows that the power law is not a plausible hypothesis for the data (p-value 0.03) (Clauset; Shalizi; Newman, 2009). The cumulative inverse distribution of the absolute values of the normalized damping coefficients is displayed in Fig. 6. Power-law decay is observed in the tail (p-value 0.82), with scaling parameter $\alpha = 5.65$. The heavy tails in the distributions of those coefficients, which describe the interactions between stocks, indicate significant variation in the strength of connections between assets.

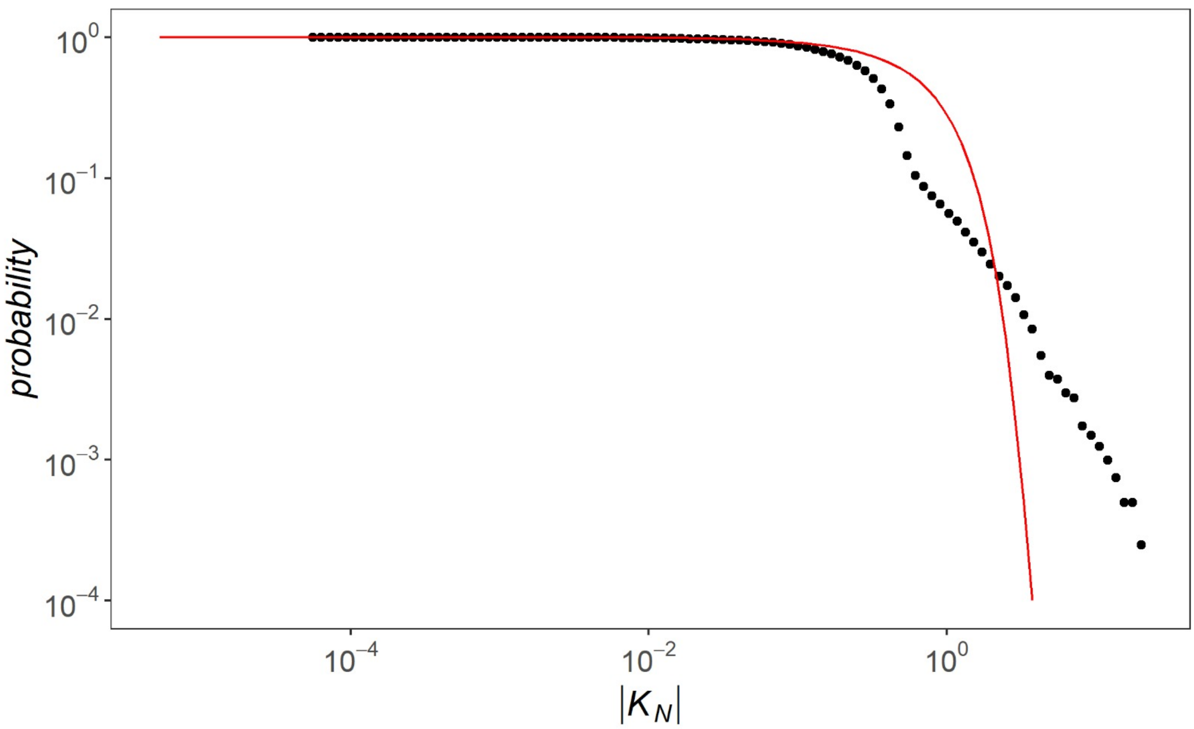

Figura 7. Dots represent the inverse cumulative distribution function of $|K_N\,(t)|$. For comparison, the red curve gives the Gaussian profile.

The same normalization method was applied to the series $K(t)$ and $U(t)$, yielding the normalized series $K_N(t)$ and $U_N(t)$. Figures 7 and 8 present the cumulative inverse distributions of the absolute values of $K_N\,(t)$ and $U_N\,(t)$. Power-law is a plausible hypothesis in both cases (p-values 0.40 and 0.41, respectively). For $|K_N\,(t)|$, the scaling parameter computed is $\alpha = 2.87$; for $|U_N\,(t)|$, we found $\alpha = 2.80$. The quantities analogous to energies defined here offer a method to consolidate information from all the assets under consideration into a single series. The heavy tails can be associated with the wide variation in the importance of external phenomena impacting the market and how its agents absorb them.

The Hurst exponent is a metric employed to quantify long-term correlations in time series (Hurst, 1951). It is commonly estimated through detrended fluctuation analysis (DFA) (Peng *et*

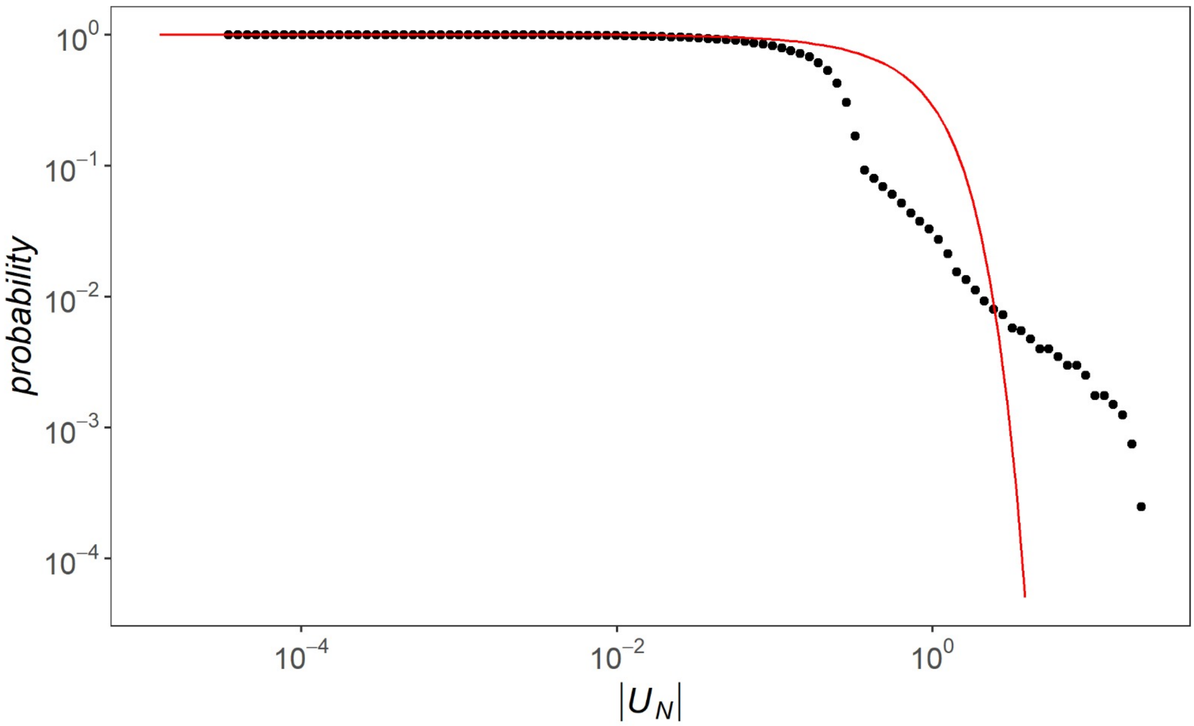

Figura 8. Dots represent the inverse cumulative distribution function of $|U_N(t)|$. For comparison, the red curve gives the Gaussian profile.

al., 1994). Persistent (or anti-persistent) series exhibit Hurst exponents above (or below) 0.5, respectively. A Hurst exponent of 0.5 indicates the absence of a trend. Both the $K(t)$ and $U(t)$ series displayed high levels of anti-persistence, characterized by Hurst exponents of $H = 0.042$ and $H = 0.044$, respectively. (they were computed with scales between 25 and $N/4$, where N is the number of points in the series). This observation aligns with the dynamics depicted in figures 3 and 4: it is noticeable that the quantities consistently revert to their baselines.

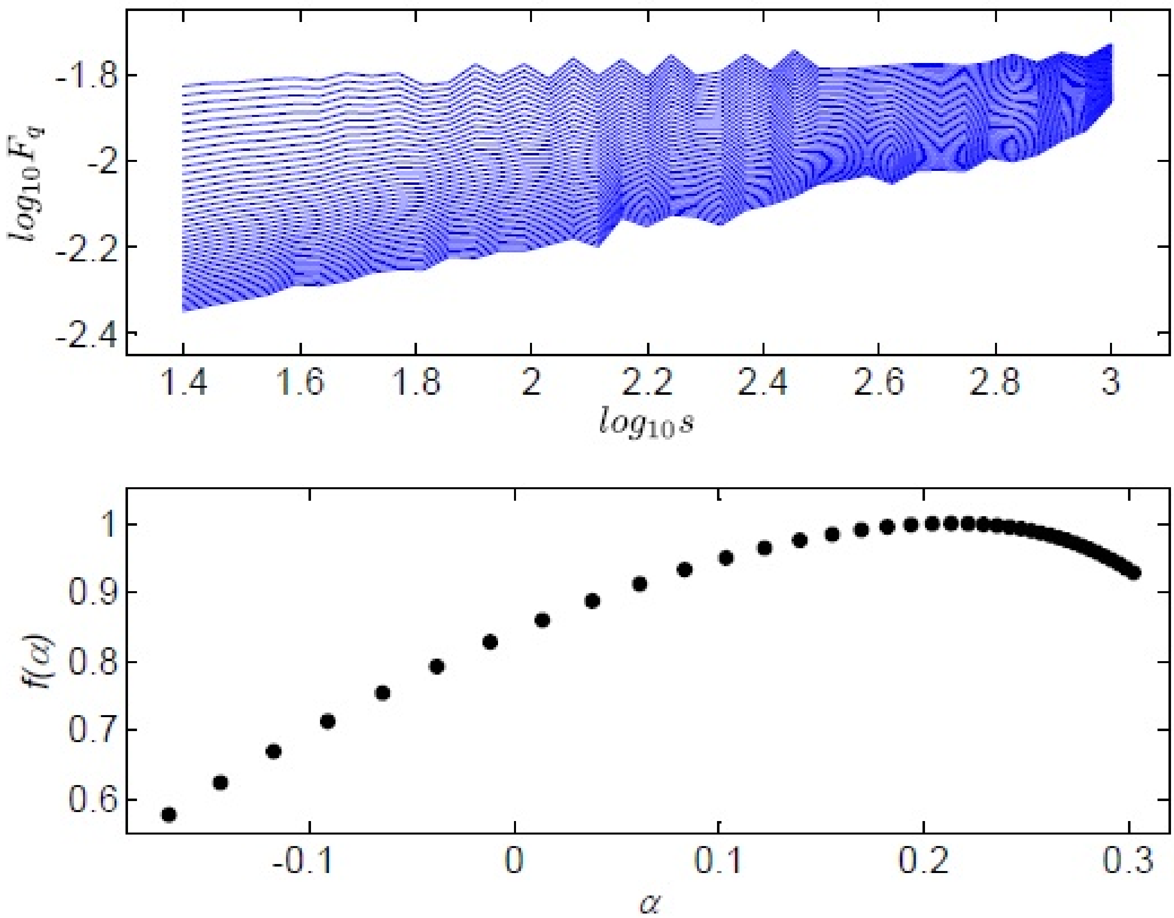

Figura 9. The top row displays, for the quantity analogous to kinetic energy, the scaling behavior of the q^{th} order fluctuation function $F_q(s)$ for 21 values of q distributed in the interval $[-2; 2]$. The singularity spectrum is exhibited in the bottom row and conforms to multifractal scaling in the data. The width, $\Delta\alpha = 0.47$, is dependent on nonlinear correlations, the distribution of the series, and its finiteness.

Generalized Hurst exponents $H(q)$ can be calculated according to the protocol described in Ref. (Kantelhardt *et al.*, 2002). The canonical Hurst exponent corresponds to $H(2)$. The significant variation of H with q denotes distinct fractal behaviors within a series, a phenomenon referred to as *multifractality*. Such variation is reflected in the singularity spectrum, where the width $\Delta\alpha$ serves as an indicator of multifractality (Kantelhardt *et al.*, 2002). This width is influenced by nonlinear correlations, the distribution of the series, and its finite nature. Shuffling a series removes the effects of correlations. Therefore, the reduction in singularity spectrum width observed after shuffling provides evidence of nonlinear correlations in the original series.

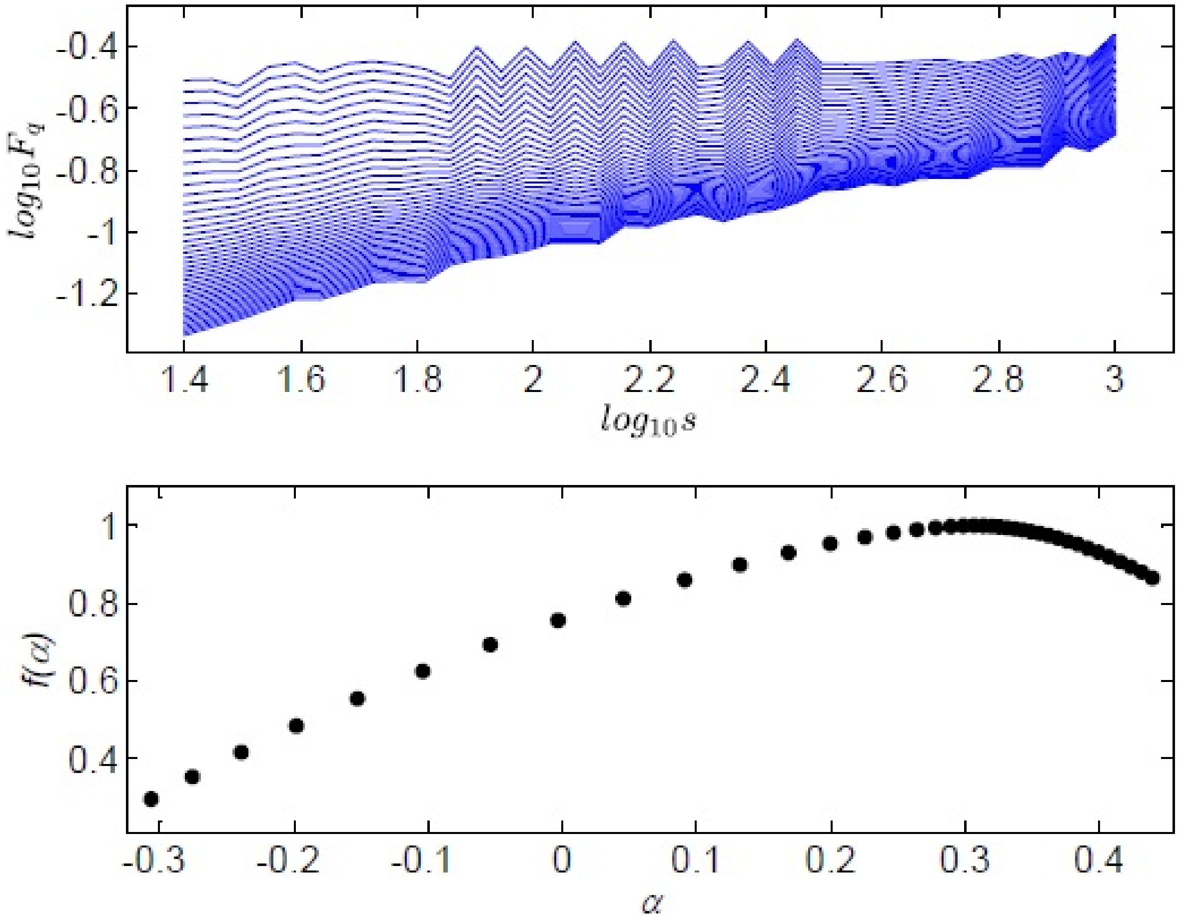

Figura 10. The top row displays, for the quantity analogous to potential energy, the scaling behavior of the q^{th} order fluctuation function $F_q(s)$ for 21 values of q distributed in the interval $[-2; 2]$. The singularity spectrum is exhibited in the bottom row and conforms to multifractal scaling in the data. The width, $\Delta\alpha = 0.75$, is dependent on nonlinear correlations, the distribution of the series, and its finiteness.

Figures 9 and 10 display the singularity spectrum for the $K(t)$ and $U(t)$ series, along with the associated scaling behavior of the q^{th} order fluctuation functions. We utilized values of q ranging from 2 to -2, as well as scales between 25 and $N/4$, where N represents the number of points in the series. We found $\Delta\alpha = 0.47$ for the original $K(t)$ series. The mean and standard deviation of $\Delta\alpha$ values for thirty samples of shuffled series were calculated as 0.31 and 0.05, respectively. Likewise, in the case of the original $U(t)$ series, $\Delta\alpha = 0.75$ was obtained. The mean and standard deviation of $\Delta\alpha$ for thirty samples of shuffled series were 0.52 and 0.07, respectively. In either case, we observe a significant decrease in width, indicating the presence of nonlinear correlations in both original series.

4. FINAL REMARKS

We proposed a model based on the assumption that the price movement of different assets is connected. As dynamic variables, we chose the logarithmic differences between asset prices and reference prices calculated through moving averages, working as proxies of the fundamental values. Dissipative coupled harmonic oscillators are the starting point for the model. The restoring forces in such a system tend to produce what in the time series literature is called mean reversion. Damping is related to the dissipation of the effects of external shocks on the economic system, which characterizes stationary series. Inertia is associated with herd behavior, which produces bubbles and crashes, and, in some cases, can be an element of non-stationarity. We got a second-order autoregressive vector model, whose estimated parameters permit us to compute quantities analogous to masses, spring constants, and friction coefficients that characterize the coupled oscillators.

We applied the model using the closing prices of 65 stocks that make up the Brazilian index Ibovespa, in the range from 12/17/2004 to 05/29/2020, totaling $4,031$ points in each series. Choosing unitary masses, we found spring constants and damping coefficients both positive and negative. Negative values of such parameters are not expected for a physical system composed of masses, springs and dampers. For an economic system, negative spring constants and damping coefficients connecting two assets may be related to opposite behaviors: the rise of one asset goes along with the fall of the other. This points to repulsive forces, which were also observed employing the PUCK model (Takayasu; Mizuno; Takayasu, 2006). The tendency of two assets to move together is associated with positive spring and damping parameters. Thus, two assets connected by particular high parameters shall be avoided in hedging strategies. All the spring constants linking a price to its reference price are positive, evidencing the importance of the regression toward the mean in the analyzed series. Positive spring constants and decay coefficients prevail over negative ones, which indicates a general trend toward a joint market movement, with damping. The distributions of the absolute values of the normalized spring constants and damping coefficients deviate substantially from the Gaussian. These distributions indicate a significant variation in the level of interdependence among stocks within the analyzed universe.

Quantities analogous to potential and kinetic energies have been defined for the economic system, enabling the aggregation of data from all explored price series. The quantity analogous to kinetic energy is, by definition, positive. Due to the possibility of positive and negative spring constants, the sign of the quantity analogous to potential energy may vary along its time evolution. However, we found it always positive. This is attributed to the predominance of positive spring constants, which are associated with the co-movement of stocks and mean reversion. Even amidst the peaks and clusters of relatively high values, both quantities ultimately gravitate back towards a region near zero. This trend is mirrored in the low values of the Hurst exponent we have uncovered, indicating anti-persistent series. Furthermore, the multifractal behavior observed in both series highlights the presence of nonlinear correlations. The dynamics of these quantities exhibit the signs of events that impacted markets around the world, such as the 2008 subprime crisis and the COVID-19 pandemic, as well as incidents related to domestic politics and the economy in Brazil. The distributions associated with the energies display a power-law decay in the tails, potentially attributable to the broad range of external influences' significance and the occasional irrational reactions of agents.

External impulses reach the economic mechanism and initiate oscillations that can vary in their regularity. While the intensity of such oscillations is directly related to that of the external stimulus, their frequencies and decay time scales are defined by the structure formed by the springs and dampers. A potential extension of this work involves leveraging this structure to simulate financial series for portfolio risk assessment purposes. Additionally, the model can be utilized for forecasting. In this scenario, parameters can be adjusted within a sliding window just before each time step to forecast price trends and volatility. Consequently, another potential avenue for this work is to explore the predictive capabilities of the model.

Acknowledgments

LVVR acknowledges the support from FAPEMIG (Grant APQ-01990-22).

STATEMENTS AND DECLARATIONS

Monique Rafaella Anunciação de Oliveira: MRAO has nothing to disclose.
Lucélia Viviane Vaz Raad: LVVR has nothing to disclose.
A. R. Bosco de Magalhães: ARBM has nothing to disclose.

5. REFERÊNCIAS

AGARWAL, G. S. Quantum Statistical Theories of Spontaneous Emission and their Relation to Other Approaches. In: ____. Berlin: Springer, 1974. (Springer Tracts in Modern Physics, v. 70).

AHN, K.; CHOI, M. Y.; DAI, B.; SOHN, S.; YANG, B. Modeling Stock Return Distributions with a Quantum Harmonic Oscillator. **Europhysics Letters**, v. 120, n. 3, p. 38003, 2017.

ATAULLAH, A.; TIPPETT, M. Equity Prices as a Simple Harmonic Oscillator with Noise. **Physica A: Statistical Mechanics and its Applications**, v. 382, p. 557–564, 2007.

ATMAN A. P. F. AMIN GONçALVES, B. Influence of the investor's behavior on the complexity of the stock market. **Brazilian Journal of Physics**, 2012. ISSN 0103-9733. Disponível em: <https://www.redalyc.org/articulo.oa?id=46423428018>.

AVRITZER, L. The Rousseff Impeachment and the Crisis of Democracy in Brazil. **Critical Policy Studies**, v. 11, p. 352–357, 2017.

Bosco de Magalhães, A. R. Searching for Robust Quantum Memories in Many Coupled Oscillators. **Physics Letters A**, v. 375, p. 4120–4129, 2011.

Bosco de Magalhães, A. R. Wealth Dynamics in a Market with Information Asymmetries. **Physical Review E**, v. 107, p. 014305, 2023.

Bosco de Magalhães, A. R.; MOKARZEL, S. G.; NEMES, M. C.; CUNHA, M. O. T. Decay Rate and Decoherence Control in Coupled Dissipative Cavities. **Physica A: Statistical Mechanics and its Applications**, v. 341, p. 234–250, 2004.

BOTTA, F.; MOAT, H. S.; STANLEY, H. E.; PREIS, T. Quantifying Stock Return Distributions in Financial Markets. **PLoS ONE**, v. 10, n. 9, p. e0135600, 2015.

BUONOCORE, R. J.; ASTE, T.; MATTEO, T. D. Measuring Multiscaling in Financial Time-Series. **Chaos, Solitons & Fractals**, v. 88, p. 38–47, 2016.

CAJUEIRO, D. O.; TABAK, B. M. The hurst exponent over time: testing the assertion that emerging markets are becoming more efficient. **Physica A: Statistical Mechanics and its Applications**, v. 336, n. 3, p. 521–537, 2004. ISSN 0378-4371.

CALDEIRA, A. O.; LEGGETT, A. J. Quantum Tunnelling in a Dissipative System. **Annals of Physics**, v. 149, p. 374–456, 1983.

CAMPBELL, J. Y.; LO, A. W.; MACKINLAY, A. C. **The Econometrics of Financial Markets**. Princeton, New Jersey: Princeton University Press, 1997.

CLAUSET, A.; SHALIZI, C. R.; NEWMAN, M. E. J. Power-Law Distributions in Empirical Data. **SIAM Review**, v. 51, n. 4, p. 661–703, 2009.

CUCINOTTA, D.; VANELLI, M. WHO Declares COVID-19 a Pandemic. **Acta Biomed.**, v. 91, p. 157–160, 2020.

DUCHA, F. A.; ATMAN, A. P. F.; Bosco de Magalhães, A. R. Information Flux in Complex Networks: Path to Stylized Facts. **Physica A: Statistical Mechanics and its Applications**, v. 566, p. 125638, 2021.

DöRFLER, F.; BULLO, F. Synchronization in Complex Networks of Phase Oscillators: A Survey. **Automatica**, v. 50, p. 1539–1564, 2014.

ENDERS, W. **Applied Econometric Time Series**. 4th. ed. [S.l.]: Wiley, 2015.

FORD, G. W.; KAC, M.; MAZUR, P. Statistical Mechanics of Assemblies of Coupled Oscillators. **Journal of Mathematical Physics**, v. 6, p. 504–515, 1965.

FRISCH, R. Propagation Problems and Impulse Problems in Dynamic Economics. In: ____. London: George Allen & Unwin, 1933. (Economic Essays in Honor of Gustav Cassel), p. 171–205.

GARCIA, M. M.; PEREIRA, A. C. M.; ACEBAL, J. L.; Bosco de Magalhães, A. R. Forecast Model for Financial Time Series: An Approach Based on Harmonic Oscillators. **Physica A: Statistical Mechanics and its Applications**, v. 549, p. 124365, 2020.

GOPIKRISHNAN, P.; PLEROU, V.; AMARAL, L. A. N.; MEYER, M.; STANLEY, H. E. Scaling of the distribution of fluctuations of financial market indices. **Phys. Rev. E**, American Physical Society, v. 60, p. 5305–5316, Nov 1999.

GRANGER, C. W. J.; NEWBOLD, P. Spurious regressions in econometrics. **Journal of Econometrics**, Elsevier BV, v. 2, n. 2, p. 111–120, jul. 1974.

HAMILTON, J. D. **Time Series Analysis**. Princeton: Princeton University Presss, 1994.

HURST, H. E. Long-term storage capacity of reservoirs. **Transactions of the American society of civil engineers**, American Society of Civil Engineers, v. 116, n. 1, p. 770–799, 1951.

KANTELHARDT, J. W.; ZSCHIEGNER, S. A.; KOSCIELNY-BUNDE, E.; HAVLIN, S.; BUNDE, A.; STANLEY, H. Multifractal detrended fluctuation analysis of nonstationary time series. **Physica A: Statistical Mechanics and its Applications**, v. 316, n. 1, p. 87–114, 2002. ISSN 0378-4371.

KULESZA, S.; BELEJ, M. Local Real Estate Markets in Poland as a Network of Damped Harmonic Oscillators. **Acta Physica Polonica A**, v. 127, n. 3-A, p. A–99–A–102, 2015.

KWAPIEŃ, J.; DROŻDŻ, S. Physical Approach to Complex Systems. **Physics Reports**, v. 515, p. 115–226, 2012.

LIDAR, D. A.; WHALEY, K. B. Decoherence-Free Subspaces and Subsystems. In: ____. Berlin: Springer, 2003. (Lecture Notes in Physics, v. 622), p. 83–120.

LUX, T.; MARCHESI, M. Scaling and Criticality in a Stochastic Multi-Agent Model of a Financial Market. **Nature (London)**, v. 397, p. 498–500, 1999.

MANTEGNA, R. N.; STANLEY, H. E. **Introduction to Econophysics: Correlations and Complexity in Finance**. Cambridge: Cambridge University Press, 1999.

MATTEO, T. D.; ASTE, T.; DACOROGNA, M. M. Long-Term Memories of Developed and Emerging Markets: Using the Scaling Analysis to Characterize their Stage of Development. **Journal of Banking & Finance**, v. 29, p. 827–851, 2005.

MENG, X.; ZHANG, J.; GUO, H. Quantum Brownian Motion Model for the Stock Market. **Physica A: Statistical Mechanics and its Applications**, v. 452, p. 281–288, 2016.

MENG, X.; ZHANG, J.; XU, J.; GUO, H. Quantum Spatial-Periodic Harmonic Model for Daily Price-Limited Stock Markets. **Physica A: Statistical Mechanics and its Applications**, v. 438, p. 154–160, 2015.

MODENESI, A. de M.; LUPORINI, V.; PIMENTEL, D. Asymmetric exchange rate pass-through: Evidence, inflation dynamics and policy implications for brazil (1999–2016). In: ____. **The Brazilian Economy since the Great Financial Crisis of 2007/2008**. Cham: Springer International Publishing, 2017. p. 69–99.

NASTASIUK, V. A. Fisher Information and Quantum Potential Well Model for Finance. **Physics Letters A**, v. 379, p. 1998–2000, 2015.

NUNES, F.; MELO, C. R. Impeachment, Political Crisis and Democracy in Brazil. **Revista de Ciencia Política**, v. 37, n. 2, p. 281–304, 2017.

ORRELL, D. A Quantum Model of Supply and Demand. **Physica A: Statistical Mechanics and its Applications**, v. 539, p. 122928, 2020.

PENG, C.-K.; BULDYREV, S. V.; HAVLIN, S.; SIMONS, M.; STANLEY, H. E.; GOLDBERGER, A. L. Mosaic organization of dna nucleotides. **Phys. Rev. E**, American Physical Society, v. 49, p. 1685–1689, Feb 1994.

RESENDE, C. C. de; PEREIRA, A. C. M.; CARDOSO, R. T. N.; Bosco de Magalhães, A. R. Investigating Market Efficiency Through a Forecasting Model Based on Differential Equations. **Physica A: Statistical Mechanics and its Applications**, v. 474, p. 199–212, 2017.

SANDERS, A. The Subprime Crisis and its Role in the Financial Crisis. **Journal of Housing Economics**, v. 17, p. 254–261, 2008.

SILVESTRE, B. S.; MONTEIRO, M. S.; VIANA, F. L. E.; SOUSA-FILHO, J. M. Challenges for Sustainable Supply Chain Management: When Stakeholder Collaboration Becomes Conducive to Corruption. **Journal of Cleaner Production**, v. 194, p. 766–776, 2018.

STROGATZ, S. H. From Kuramoto to Crawford: Exploring the Onset of Synchronization in Populations of Coupled Oscillators. **Physica D: Nonlinear Phenomena**, v. 143, p. 1–20, 2000.

TAKAYASU, M.; MIZUNO, T.; TAKAYASU, H. Potential Force Observed in Market Dynamics. **Physica A: Statistical Mechanics and its Applications**, v. 370, p. 91–97, 2006.

TSAY, R. S. **Analysis of Financial Time Series**. 3rd. ed. [S.l.]: Wiley, 2010.

XAVIER, P. O. C.; ATMAN, A. P. F.; Bosco de Magalhães, A. R. Equation-Based Model for the Stock Market. **Physical Review E**, v. 96, p. 032305, 2017.

YANG, G.; ZHU, C. G.; AN, K. N.; HUANG, J. P. Overall Fluctuations and Fat Tails in an Artificial Financial Market: The Two-Sided Impact of Leveraged Trading. **Physics Letters A**, v. 379, p. 1857–1863, 2015.

YE, C.; HUANG, J. P. Non-Classical Oscillator Model for Persistent Fluctuations in Stock Markets. **Physica A: Statistical Mechanics and its Applications**, v. 387, p. 1255–1263, 2008.